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Proceedings of a

Conference on the Theoretical Aspects of

CONTROLLED FUSION RESEARCH

Gatlinburg, Tennessee
APRIL 27–28, 1959

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OAK RIDGE NATIONAL LABORATORY
Oak Ridge, Tennessee

FOREWORD

Once again the resort city of Gatlinburg, Tennessee, was the scene of a meeting devoted to Project Sherwood (controlled fusion research) when a small group of theorists (and a few experimentalists) met there on April 27 and 28, 1959. Unlike the previous meeting here, which was in June of 1956 and which covered all aspects of the Project, this meeting was limited to the subject of "Theoretical Aspects of Controlled Fusion Research." The other striking difference in the two meetings was a more relaxed attitude as a result of the complete declassification of the project in the interim. Unfortunately, the difficulties of nature are not legislated away as easily, and the contents of the papers reflect the degree of concern with the various plasma misbehaviors.

The present report represents a compilation of the papers presented at this conference. Some authors have chosen to submit only the abstract of their talks since they plan to publish in the open literature shortly.

The conference committee wishes to express its thanks to Mr. D. D. Cowen of ORNL who, with his staff, was responsible for the smooth handling of the local arrangements. We also owe a debt of gratitude to Mrs. Lorraine Abbott for her expert advice and assistance in preparing this report.

Committee:

A. Simon, Chairman
R. G. Alsmiller, Jr.
T. K. Fowler
E. G. Harris

CONTENTS

A. TRANSPORT PHENOMENA

THERMALIZATION OF A FAST ION IN A PLASMA*

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New York University

Abstract

A fast ion is injected into a plasma in equilibrium. We determine the time history of the probability distribution of this ion in velocity space. This is done by numerical integration of the linearized, space-independent Fokker-Planck equation with both the ion-ion and ion-electron terms retained. The mean time of thermalization is calculated for several widely separated injection velocities.

Suppose a single ion of velocity $\vec{\xi}_0$ is injected at time $t = 0$ into a homogeneous plasma in thermal equilibrium with no external electromagnetic fields present. The probability distribution $f(\vec{\xi}, t)$ of this ion in velocity space satisfies the space-independent Fokker-Planck equation[1]

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial \xi_r}(a_r f) + \frac{\partial^2}{\partial \xi_r \partial \xi_s}\left(\frac{1}{2} b_{rs} f\right) \tag{1}$$

and the initial condition

$$f(\vec{\xi}, 0) = \delta(\vec{\xi} - \vec{\xi}_0) \quad . \tag{2}$$

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* The work presented in this paper is supported by the AEC Computing and Applied Mathematics Center, Institute of Mathematical Sciences, New York University, under Contract AT(30-1)-1480 with the U. S. Atomic Energy Commission.

1. See Rosenbluth, MacDonald and Judd, Phys. Rev., 107, 1 (1957), or Grad, Thermonuclear Reaction Rates in an Electrical Discharge, NYO-7977, Inst. of Math. Sciences, N. Y. Univ., Jan. 1958.

The values of the dynamical friction coefficient a_r and the dispersion coefficient b_{rs} to be taken in (1) are those corresponding to the (Maxwellian) distributions of ions and electrons in the original plasma.[2]

We introduce the dimensionless independent variables

$$\vec{x} = \vec{\xi}/(RT)^{1/2} \quad , \tag{3}$$

$$\tau = \frac{4\pi n e^4 \ln\Lambda}{M^{1/2}(kT)^{3/2}} t \quad , \tag{4}$$

where T is the plasma temperature (assumed the same for electrons or ions), M is the ion mass, k is Boltzmann's constant, $R = k/M$, e is the electronic charge, n the number density of ions or electrons, and

$$\Lambda = \frac{3}{2} \frac{(kT)^{3/2}}{\sqrt{\pi n}\, e^3} \tag{5}$$

is the ratio of the Debye length to the mean distance of closest approach in a Coulomb encounter. (All quantities are expressed in cgs electrostatic units.) Then equations (1) and (2) become conditions of the form

$$\frac{\partial \phi}{\partial \tau} = - \frac{\partial}{\partial x_r}(\alpha_r \phi) + \frac{\partial^2}{\partial x_r \partial x_s}\left(\frac{1}{2}\beta_{rs}\phi\right) \quad , \tag{6}$$

$$\phi(\vec{x}, 0) = \delta(\vec{x} - \vec{x}_o) \quad , \quad \vec{x}_o = \vec{\xi}_o/(RT)^{1/2} \quad , \tag{7}$$

on the dimensionless probability distribution

$$\phi(x, \tau) = (RT)^{3/2} f(\xi, t) \quad . \tag{8}$$

Our ultimate goal is to numerically integrate (6), (7), a system involving two velocity dimensions and one time dimension. Here we assume spherical symmetry in velocity space. If we are interested in following only the speed of the injected test ion, this will be an excellent approximation.

Hence we set

$$\frac{1}{x^2} \iint_{|y|=x} \phi(\vec{y}, t)\, d\omega_y = g(x, t) \tag{9}$$

and obtain for g the differential equation

$$\frac{\partial g}{\partial \tau} = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{x^2} \frac{\partial}{\partial x}\left(xGg + G\frac{\partial g}{\partial x}\right) \tag{10}$$

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2. See Grad, op. cit., for the precise form of these coefficients in this case.

and the initial condition

$$g(x, 0) = x_o^{-2} \delta(x - x_o) \quad . \tag{11}$$

The function $G = G(x)$ is defined as

$$G(x) = F(x) + \rho F(\frac{x}{\rho}) \quad , \tag{12}$$

where (at least for singly charged ions)

$$\rho^2 = M/m \tag{13}$$

is the ratio of the ion to the electron mass and

$$F(x) = \frac{1}{x} \int_o^x e^{-y^2/2} dy - e^{-x^2/2} \tag{14}$$

We proceed to the numerical solution of (10), (11) by finite differences. We approximate (10) by an explicit difference equation; i. e., we take a rectangular mesh with a spacing Δx in velocity and $\Delta\tau$ in time and replace $\frac{\partial g}{\partial \tau}$ by a forward difference. To reduce truncation error, the right-hand side of (10) is first expanded into the form

$$\begin{aligned} \frac{\partial g}{\partial \tau} &= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{x^2} [(xG' + G)g + (G' + xG) \frac{\partial g}{\partial x} + G \frac{\partial^2 g}{\partial x^2}] \\ &= C(x)g + A(x) \frac{\partial g}{\partial x} + B(x) \frac{\partial^2 g}{\partial x^2} \quad . \end{aligned} \tag{15}$$

The x-derivatives of g in (15) are then replaced by centered differences, while the coefficients A, B, C are evaluated analytically. Because $A(x)$ becomes infinite at $x = 0$, the first velocity mesh point is taken at $x = \frac{1}{2}\Delta x$; differences centered at this point are computed by assuming g to be an even function of x. The truncation error of this scheme is of the order of $(\Delta t)^2$ or $(\Delta x)^4$.

Since the maximum of $x^{-2}G$ occurs at $x = 0$ and is equal to $\frac{1}{3}$, the Courant-Friedrichs-Lewy stability criterion is satisfied if

$$\lambda = \frac{\Delta\tau}{(\Delta x)^2} \leq \frac{3}{2}\sqrt{\frac{\pi}{2}} \approx 1.88. \tag{16}$$

For safety, we choose $\lambda = 1.75$.

In most of the cases considered below, the initial values (11) of g are represented numerically by choosing x_o to coincide with a mesh point and taking $g(x, 0)$ to be $x_o^{-2}(\Delta x)^{-1}$ at this mesh point and zero at the others. In the one case where $x_o > \rho$, this prescription would cause oscillations which reach unacceptably large amplitudes before damping out. The initial values of g in this case are chosen to be the values g would have at some later time $\hat{\tau}$ (small relative to the time scale of thermalization) if the coefficients A, B, C in (15) had everywhere the values they have at x_o:

$$g(x, 0) = \frac{\exp[C\hat{\tau} - (x - x_o + A\hat{\tau})^2/4B\hat{\tau}]}{2x_o^2(\pi B\hat{\tau})^{1/2}} \tag{17}$$

The choice of numerical upper bounds for x and τ is facilitated by the fact that the exact solution of (10), (11) approaches as $\tau \to \infty$ the Maxwellian distribution

$$g_M(x) = \sqrt{\frac{2}{\pi}}\, e^{-x^2/2} \quad . \tag{18}$$

Thus for an x not much larger than x_o the solution remains altogether negligible for all time. We take such an x for an outer boundary and impose there the simple boundary condition $g = 0$. The upper bound on τ is determined by the computation itself: we stop computing whenever $g(x, \tau)$ is as close to $g_M(x)$ as the numerical approximation allows.

Numerical computations have been performed for a plasma consisting of deuterium, for which $\rho = 60.5948$. Four widely spaced injection velocities x_o were chosen: $x_o = 0$, $x_o = 1.55$, $x_o = 9.7$, and $x_o = 240$. The first of these was chosen to provide a reference relaxation time. The second corresponds to the center of the initial distribution used by Rosenbluth.[3] The choice of the last two can be most easily understood by reference to the "friction curve" — the plot of the total friction coefficient α (cf. (6)) as a function of x. This curve begins at zero (for $x = 0$), rises to a maximum value α_M at about $x = 1.3$, decreases to a minimum between $x = 1.3$ and $x = \rho$, rises to a second maximum value $\frac{1}{2}\alpha_M$ at $x = 1.3\rho$, and finally falls off toward zero in proportion to x^{-2}. The velocity $x = 9.7$ lies at the central minimum, while $x = 240$ is approximately that velocity for which α reaches on its final downward curve the same value as it has at $x = 9.7$.

The case $x_o = 240$ requires smoothing of the initial data. The time $\hat{\tau}$ of smoothing (see (17)) was taken as 37.7.

The numerical parameters used in the four cases are summarized in Table I.

In the three cases where x_o was positive, the solutions behave quite similarly (see Fig. 1). At first, the initial delta-function diffuses into a Gaussian whose width is determined by the dispersion coefficient B(x) and whose peak moves toward lower velocities at the rate A(x). This regular evolution continues until the inner tail of the Gaussian reaches $x = 0$. (Fig. 1C). At that time, a second peak forms at $x = 0$ (Fig. 1D). This peak

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3. MacDonald, Rosenbluth, and Chuck, Phys. Rev., 107, 351 (1957).

Table I — Details of the numerical computation

Injection velocity x_o	0	1.55	9.7	240
Smoothing time $\hat{\tau}$	0	0	0	37.7
Velocity mesh Δx	0.1	0.05	0.06	0.1
Time mesh $\Delta\tau$	0.0175	0.004375	0.0063	0.0175
Upper velocity bound x_{max}	7.0	7.0	12.0	250

rapidly increases in height and width (Fig. 1E), completely swallowing up the original Gaussian, until (Fig. 1F) it somewhat surpasses the Maxwellian distribution (18). Finally (Fig. 1G) it slowly falls back and broadens, approaching (18) exponentially.

When $x_o = 0$, the initial distribution is just an extreme case of the pattern of Figure 1F. Hence an exponential decay toward the Maxwellian distribution begins immediately.

Some of the quantitative details of this general picture are given in Table II. In the case $x_o = 240$, the smoothing time $\hat{\tau}$ is included in all elapsed times. A physical idea of the size of the units involved may be ob-

Table II

Injection velocity x_o	0	1.55	9.7	240
Time τ_1, at which central peak begins to form	—	0.9	50	1840
Position x of outer peak at this time	—	1.07	7.08	11.75
Time τ_2, at which outer peak disappears	—	1.3	100	1970
Time τ_3, at which exponential decay begins	0	3.9	261	2140
Time constant $\tilde{\tau}$ of exponential decay	3	8	15	20
Mean thermalization time $\tau_4 = \tau_3 + \tilde{\tau}$	3	12	276	2160
Ratio of mean thermalization time to Spitzer collision time	~1	~3	~75	~600

These computations were performed on the IBM 704 at the Institute of Mathematical Sciences, New York University.

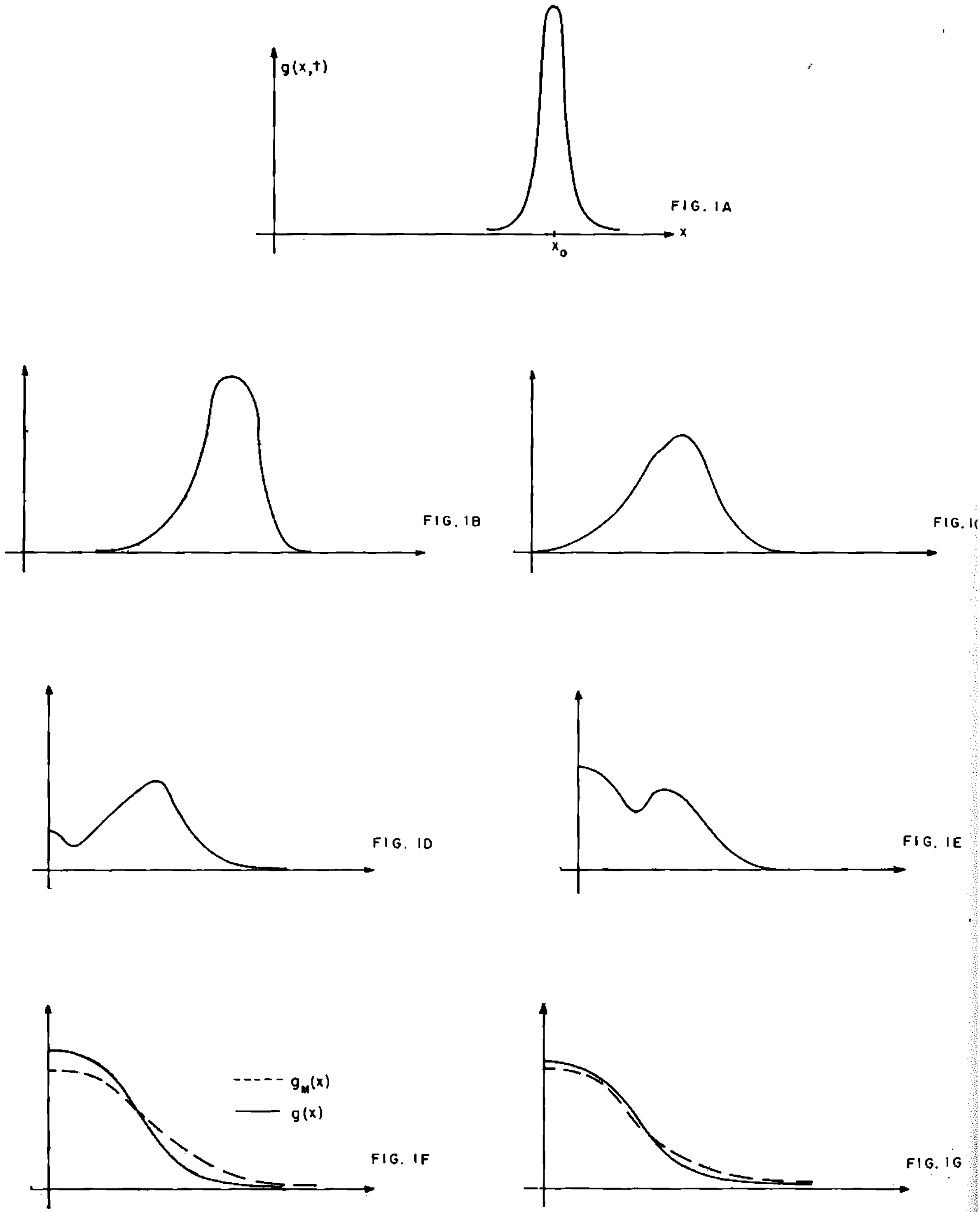

FIG. 1. SUCCESSIVE STAGES OF THERMALIZATION

tained by noting that the root mean square plasma ion speed corresponds to $x = \sqrt{3} \approx 1.73$, while the r. m. s. electron speed corresponds to $x = \rho\sqrt{3} \approx 105$. Furthermore, the Spitzer ion-ion collision time[4]

$$t_c = M^{1/2}(3kT)^{3/2}/8 \times 0.714\pi ne^4 \ell n\Lambda \tag{19}$$

is equal to 3.62 units of tau.

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4. Spitzer, Physics of Fully Ionized Gases, Interscience, New York, 1955.

A VARIATIONAL CALCULATION OF PLASMA TRANSPORT PROPERTIES

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and

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Abstract

A variational principle is given for the electrical conductivity of a fully ionized plasma. Use of a very simple trial function yields Spitzer's value to within 2%. The method can be generalized so as to apply to all transport coefficients.

The Fokker-Planck Equation

The distribution function f, describing the joint distribution in position and velocity of electrons colliding with a locally equal number of infinitely massive protons in an external electric field $\underline{\mathcal{E}}$, consistently neglecting magnetic effects[1], is determined by the Fokker-Planck equation

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \frac{\partial f}{\partial \underline{r}} + \frac{e}{m} \underline{\mathcal{E}} \cdot \frac{\partial f}{\partial \underline{v}} = - \frac{\partial}{\partial \underline{v}} \cdot \underline{j} . \qquad (1)$$

The velocity space current density $\underline{j}$ is given by

1. M. N. Rosenbluth, Wm. M. MacDonald, and D. L. Judd, Phys. Rev. 107, 1 (1957).

$$-\underline{j} = \frac{4\pi e^4 \ln\Lambda}{m^2}\Big\{-f(\underline{v})\frac{\partial}{\partial \underline{v}}\int d^3v' \frac{f(\underline{v}')}{|\underline{v}-\underline{v}'|}$$

$$+\frac{1}{2}\frac{\partial f(\underline{v})}{\partial \underline{v}}\cdot\frac{\partial^2}{\partial \underline{v}\,\partial \underline{v}}\int d^3v' f(\underline{v}')\,|\underline{v}-\underline{v}'| \tag{2}$$

$$+\frac{1}{2}\frac{\partial f(\underline{v})}{\partial \underline{v}}\cdot\frac{v^2\,\underline{I}-\underline{v}\,\underline{v}}{v^3}\Big\},$$

$$\Lambda = 12\pi N\left(\frac{kT}{4\pi N e^2}\right)^{3/2}$$

It is convenient to transform the above expression to a more symmetric form by integration by parts. Namely since

$$\frac{\partial}{\partial \underline{v}}\int d^3v' \frac{f(\underline{v}')}{|\underline{v}-\underline{v}'|}$$

$$= -\frac{1}{2}\int d^3v' f(\underline{v}')\frac{\partial}{\partial \underline{v}'}\nabla_{v'}^{2}\,|\underline{v}-\underline{v}'| \tag{3}$$

$$= \frac{1}{2}\int d^3v' \frac{\partial f(\underline{v}')}{\partial \underline{v}'}\cdot\frac{\partial^2}{\partial \underline{v}'\,\partial \underline{v}'}\,|\underline{v}-\underline{v}'|\;,$$

on defining

$$\underline{g} = \underline{v}-\underline{v}'\;, \tag{4}$$

one can write eq. (2) in the form

$$-\underline{j} = \frac{2\pi e^4 \ln\Lambda}{m^2}\Big\{\int d^3v'\left[f(\underline{v}')\frac{\partial f(\underline{v})}{\partial \underline{v}} - f(\underline{v})\frac{\partial f(\underline{v}')}{\partial \underline{v}'}\right]\cdot\frac{g^2\,\underline{I}-\underline{g}\,\underline{g}}{g^3} \tag{5}$$

$$+\frac{\partial f(\underline{v})}{\partial \underline{v}}\cdot\frac{v^2\,\underline{I}-\underline{v}\,\underline{v}}{v^3}\Big\}.$$

Quasi-equilibrium

Assume that there is spatial homogeneity and that there is only a uniform external electric field $\underline{\mathcal{E}}$. Then if $\underline{\mathcal{E}}$ is treated as a perturbation, in terms of a quantity ϕ, of the same order as $\underline{\mathcal{E}}$, one writes

$$f = f_o(1+\phi)\;, \tag{6}$$

where

$$f_o = N(m/2\pi kT)^{3/2} e^{-mv^2/2kT} . \tag{7}$$

Then, on linearization[2] of eq. (1), there results

$$\frac{e}{m} \underline{\mathcal{E}} \cdot \frac{\partial f_o}{\partial \underline{v}} = - \frac{e}{kT} \underline{\mathcal{E}} \cdot \underline{v} f_o = K\phi . \tag{8}$$

The linear operator K is defined by

$$K\phi = \frac{2\pi e^4 \ln\Lambda}{m^2} \frac{\partial}{\partial \underline{v}} \cdot \left\{ \int d^3v' f_o(\underline{v}) f_o(\underline{v}') \left[\frac{\partial\phi(\underline{v})}{\partial \underline{v}} - \frac{\partial\phi(\underline{v}')}{\partial \underline{v}'} \right] \cdot \frac{g^2 \underline{I} - \underline{g}\underline{g}}{g^3} + f_o(\underline{v}) \frac{\partial\phi(\underline{v})}{\partial \underline{v}} \cdot \frac{v^2 \underline{I} - \underline{v}\underline{v}}{v^3} \right\} . \tag{9}$$

Define the inner product $(\psi, K\phi)$ by

$$(\psi, K\phi) = \int d^3v \, \psi(\underline{v}) K\phi . \tag{10}$$

If one takes the expression obtained by employing eq. (9) directly with eq. (10), interchanges $\underline{v}$ and $\underline{v}'$, and then forms one half the sum of the former and latter expressions, there results

$$(\psi, K\phi) = - \frac{\pi e^2 \ln\Lambda}{m^2} \left\{ \int d^3v d^3v' \left[\frac{\partial\psi(\underline{v})}{\partial \underline{v}} - \frac{\partial\psi(\underline{v}')}{\partial \underline{v}'} \right] \left[\frac{\partial\phi(\underline{v})}{\partial \underline{v}} - \frac{\partial\phi(\underline{v}')}{\partial \underline{v}'} \right] : \frac{g^2 \underline{I} - \underline{g}\underline{g}}{g^3} + \int d^3v f_o(\underline{v}) \frac{\partial\psi(\underline{v})}{\partial \underline{v}} \frac{\partial\phi(\underline{v})}{\partial \underline{v}} : \frac{v \, \underline{I} - \underline{v}\underline{v}}{v^3} \right\} . \tag{11}$$

It is obvious from eq. (11) that K is a symmetric operator, i.e.

$$(\psi, K\phi) = (K\psi, \phi) , \tag{12}$$

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2. S. Chapman and T.G. Cowling - The Mathematical Theory of Non-Uniform Gases, Second Edition, Cambridge, 1953.

and moreover that K is a negative operator, i.e.

$$(\phi, K\phi) \leq 0 \tag{13}$$

This latter conclusion follows immediately from the observation that for an arbitrary vector $\underline{a}$

$$\underline{a}\,\underline{a} : (g^2\underline{I} - \underline{g}\,\underline{g})/g^2 = a^2 - (\underline{a}\cdot\underline{g}/g)^2 \geq 0 \tag{14}$$

Electrical Conductivity

Multiply eq. (8) by ϕ and integrate with respect to $\underline{v}$. Observe that to lowest order in the parameter of smallness

$$\begin{aligned} <\underline{v}> &= \int d^3v\,\underline{v}\,f/N \\ &= \int d^3v\,\underline{v}\,f_o\,\phi/N\ . \end{aligned} \tag{15}$$

Therefore,

$$\frac{Ne}{kT}\,\underline{\mathcal{E}}\cdot <\underline{v}> = -(\phi, K\phi). \tag{16}$$

Now it follows from the form of eq. (8) that ϕ must have the form[2]

$$\phi(\underline{v}) = \frac{\underline{\mathcal{E}}}{\mathcal{E}}\cdot\frac{\underline{v}}{v}\,\Phi(v)\ , \tag{17}$$

whence

$$\begin{aligned} N<\underline{v}> &= \int d^3v\,f_o(v)\,\Phi(v)\,\frac{\underline{v}\,\underline{v}}{v}\cdot\frac{\underline{\mathcal{E}}}{\mathcal{E}} \\ &= \frac{1}{3}\,\frac{\underline{\mathcal{E}}}{\mathcal{E}}\int d^3v\,v\,f_o(v)\,\Phi(v)\ , \end{aligned} \tag{18}$$

that is the mean velocity lies along the electric field. Thus the electrical conductivity can be written, on employing eq. (16)

$$\sigma = \frac{Ne\,|<\underline{v}>|}{\mathcal{E}} = -\frac{kT}{\mathcal{E}^2}(\phi; K\phi) = -\frac{kT}{\mathcal{E}^2}(\psi, \phi) \tag{19}$$

Variational Principle

Define

$$\psi(\underline{v}) = -\frac{e}{kT}\underline{\mathcal{E}}\cdot\underline{v}\, f_o(\underline{v}) \tag{20}$$

$$\lambda\{\phi\} = \frac{kT}{\mathcal{E}^2}\,\frac{(\psi,\phi)^2}{-(\phi,K\phi)} \tag{21}$$

Note that

$$\lambda\{\phi+\Delta\phi\}-\lambda\{\phi\}$$

$$= -\frac{kT}{\mathcal{E}^2}\left\{\frac{(\psi,\phi)^2+2(\psi,\Delta\phi)(\psi,\phi)+(\psi,\Delta\phi)^2}{(\phi,K\phi)+2(\Delta\phi,K\phi)+(\Delta\phi,K\Delta\phi)}-\frac{(\psi,\phi)^2}{(\phi,K\phi)}\right\}$$

$$= -\frac{kT/\mathcal{E}^2}{[\phi+\Delta\phi,K(\phi+\Delta\phi)]}\left\{2\left[\Delta\phi,\psi-\frac{(\psi,\phi)^2}{(\phi,K\phi)}K\phi\right](\psi,\phi)\right.$$

$$\left.-\frac{(\psi,\phi)^2}{(\phi,K\phi)}(\Delta\phi,K\Delta\phi)+(\psi,\Delta\phi)^2\right\} \tag{22}$$

Thus the condition that λ be stationary is

$$\psi = \frac{(\psi,\phi)^2}{(\phi,K\phi)}K\phi\ . \tag{23}$$

Given any solution of eq. (23) one can always find a constant c such that $\phi = c\phi$ satisfies

$$\psi = K\phi' , \tag{24}$$

which is just eq. (8). This renormalization clearly does not change the value of λ. Moreover when ϕ satisfies eq. (23) (or ϕ eq. (24))

$$\lambda\{\phi\} = -\frac{kT}{\mathcal{E}^2}(\psi,\phi) = \sigma \tag{25}$$

Thus λ is stationary for variations in ϕ about that function which satisfies eq. (24), and moreover the associated external value of λ is just the desired conductivity σ.

Suppose that ϕ satisfies eq. (23) and that $\Delta\phi$ is an arbitrary and not necessarily small variation. Then it follows from eq. (22) that

$$\lambda \{\phi + \Delta\phi\} - \lambda \{\phi\}$$

$$= \frac{kT/\mathcal{E}^2}{[\phi + \Delta\phi,\ K(\phi + \Delta\phi)]} \left\{ (\Delta\phi,\ K\phi)^2 - (\phi,\ K\phi)(\Delta\phi,\ K\Delta\phi) \right\} \tag{26}$$

In order to determine the sign of the quantity in curly braces observe that since K is a negative operator,

$$o \leq (\phi,\ K\phi)[\Delta\phi + x\phi,\ K(\Delta\phi + x\phi)]$$

$$\leq (\phi,\ K\phi)[(\Delta\phi,\ K\Delta\phi) + 2x(\Delta\phi,\ K\phi) + x2(\phi,\ K\phi)] \tag{27}$$

$$\leq (\phi,\ K\phi)^2 \left[x + \frac{(\Delta\phi,\ K\phi)}{(\phi,\ K\phi)}\right]^2 + (\phi,\ K\phi)(\Delta\phi,\ K\Delta\phi) - (\Delta\phi,\ K\phi)^2 ,$$

where x is arbitrary.

Therefore

$$o \leq (\phi,\ K\phi)(\Delta\phi,\ K\Delta\phi) - (\Delta\phi,\ K\phi)^2 \tag{28}$$

and

$$\lambda \{\phi + \Delta\phi\} - \lambda\{\phi\} \leq o .$$

Thus the variational principle is an absolute maximum principle, and the resultant extremum unique.

The conductivity of a plasma has been computed by Spitzer and Härm[3] by numerical integration of eq. (8). The results are reported in terms of σ_L, the conductivity of a Lorentzian gas, namely a plasma in which the electrons collide only with infinitely massive positive ions, and

$$\sigma_L = \frac{2^{5/2}}{\pi^{3/2}} \frac{1}{\ln \Lambda} \left(\frac{kT}{m}\right)^{1/2} \frac{kT}{e^2} \tag{29}$$

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3. L. Spitzer and R. Härm, Phys. Rev. <u>89</u>, 977 (1953).

Thus

$$\frac{\sigma_{Spitzer}}{\sigma_L} = 0.582, \tag{30}$$

while the trial functions $\phi = \underline{\mathcal{E}} \cdot \underline{v}\, v/$ and $\phi = \underline{\mathcal{E}} \cdot \underline{v}\, v^2/\varepsilon$ yield respectively

$$\frac{\sigma}{\sigma_L} = 0.569$$

$$\frac{\sigma}{\sigma_L} = 0.540 \tag{31}$$

The variational principle can be generalized to an arbitrary mixture of neutral and charged particles in a magnetic field and has been employed for this purpose by Walter Marshall[4], using the usual two body collision integrals rather than the Fokker-Planck equation.

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4. Walter Marshall - The Kinetic Theory of an Ionized Gas, A. E. R. E. T/R 2247, 2248, 2419, Atomic Energy Research Establishment, Harwell, Berkshire, 1958.

TEST PARTICLES IN A PLASMA

M. N. Rosenbluth
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Abstract

A charged particle is considered to move in a preassigned orbit. The plasma is treated as a fluid, i.e., a medium in which $e \to 0$, $m \to 0$, $n \to \infty$ such that e/m and ne remain constant. The plasma becomes polarized so that there is a cloud of charge around the test particle. The test particle is not at the electrical center of the cloud, so that there is an electric field acting on it. This procedure gives the usual frictional drag, except for the proper effective mass. There is an additional drag due to the emission of plasma waves. By considering a Maxwell distribution of "test particles," the total plasma wave emission is calculated.

The test particle problem has been solved in the absence and in the presence of a constant magnetic field. With a magnetic field, the drag parallel to the field resembles the zero field case, except that the Larmor radius may replace the Debye length in the long wavelength cutoff. The drag perpendicular to the field has no counterpart in the zero field case. It exhibits some qualitatively new features that are due to resonant interactions with field particles.

KINETIC EQUATIONS FOR A PLASMA

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Abstract

If in the Liouville equation, the coordinates of all particles but one, but two, etc., are integrated out, one obtains a chain of equations for the one-, two-, etc., body distributions. The chain can be solved rigorously by expanding in powers of the charge. The lowest order means the limit $e \to 0$, $m \to 0$, $n \to \infty$, such that e/m and the ne remain constant. In this case the particles are independent, and the one-body distribution obeys the collisionless Boltzmann equation. In the next order the solutions for the n-body functions can be expressed in terms of two-body correlation functions.

If no particles are distinguished the equation for the one-body distribution is the Boltzmann equation. If one particle is distinguished the symmetry of the density in phase space must be reduced. The equation for the distinguished particle is the Fokker-Planck equation. The test-particle problem is an incosistent approximation which is first order in the charge of the distinguished particle and zero order in the other charges.

The consistent test-particle problem is formulated and solved for the case of zero external magnetic field. The resulting Fokker-Planck equation contains new terms that arise from the emission of plasma waves.

B. WAVES IN PLASMAS

MICROWAVE EMISSION FROM HIGH TEMPERATURE PLASMAS*

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Abstract

The emission of cyclotron radiation from near-relativistic plasma electrons has been estimated. The calculation presented here takes into account the relativistic effects on the frequencies radiated by energetic electrons. Radiation parallel to the confining magnetic field is broadened by the Doppler effect or the relativistic mass dependence on the electron energy. Radiation perpendicular to the field is in a broad distorted line due to the relativistic mass variation. Radiation is also emitted in higher harmonics of the fundamental cyclotron frequency due to the asymmetry in the laboratory frame of the electric field resulting from the electron charge. The emission has been calculated by integration over the velocity spectrum of a Maxwell-Boltzmann distribution in electron velocity and summed over all the contributing harmonics. The results of Trubnikov and Kudryavtsev, reported at the September, 1958, Geneva Conference, are roughly substantiated.

Aside from its obvious application as a diagnostic tool some of the current interest in cyclotron radiation emitted from hot plasma stems from a recent prediction[1] that it would amount to a serious energy loss. The calculation I wish to report on estimated the plasma cyclotron emission by calculating the index of refraction and absorption coefficient of the plasma. From these optical constants the absorption of incident radiation was determined and by invoking Kirchhoff's relation the emission was found. The main feature of the calculation was that the variation in resonant frequency due to the relativistic

*This report differs from the talk presented at the meeting. In the integration over velocity space an unfortunate error in sign was made with the result that the emission was greatly underestimated. Luckily, I. Bernstein and M. Rosenbluth had been looking into the problem (See discussion.) and recognized that an error had been made in the integration. The author is deeply indebted to them both for kindly calling his attention to the error and preventing its further propagation.

1. B. A. Trubnikov and V. S. Kudryavtsev, Second United Nations Conference on Peaceful Uses of Atomic Energy, A/conf 15, p. 2213 (1958).

behavior of the electrons was taken into account. The electrons resonated at differing frequencies with the result that the phase of an incident wave was simultaneously advanced by some electrons and retarded by others. Thus, the index of refraction for all the electrons was greatly reduced. The absorption or emission occurs in a greatly broadened frequency interval and diminished line height.

Unfortunately, the only generally available account of Trubnikov and Kudryavtsev's work is rather skimpy in detail of just how the broad distribution of emission with frequency was obtained from an integral over velocity space. To better understand the Russian work I have decided not to attempt to summarize my own lengthy calculation in a fifteen-minute paper, but instead to start off by computing the emission spectrum directly. There are two independent sources of line broadening, one due to the collision frequency of the electrons with the ions and other electrons and the second due to the relativistic change in resonant frequency. These two line profiles must be folded together. The relativistic spread in non-relativistic approximation is assumed to be given by a Maxwell-Boltzmann distribution in velocity space. For radiation perpendicular to the magnetic field (also in the case of a mirror machine with small field gradient along the field) the electron velocity component perpendicular to the field is of interest

$$N(V_p)\,dV_p = N_0 \frac{m_0c^2}{2kT} \exp -\left[\frac{m_0c^2}{2kT}\frac{V_p^2}{c^2}\right] d\left(\frac{V_p^2}{c^2}\right) \tag{1}$$

The frequency shift of $\omega_0 = (eH/m_0c)$ is due to the relativistic change in mass of the electron, i. e., $\Delta\omega_0/\omega_0 = v^2/c^2$.

For radiation parallel to the magnetic field taken to be along the Z axis, when large electron velocities parallel to the magnetic field occur, the Doppler shift $\Delta\omega_0/\omega_0 \sim v_z/c$ is of interest.

$$N(V_z)\,dV_z = N_0 \sqrt{\frac{m_0c^2}{2\pi kT}} \exp -\left[\frac{m_0c^2}{2kT}\frac{V_z^2}{c^2}\right] d\left(\frac{V_z}{c}\right) \tag{2}$$

Particularly in a magnetic mirror geometry and in any geometry for frequencies above the fundamental frequency Eq. 1 is of primary interest. The resultant line profile is akin to a Voigt profile:

$$I(\omega) \sim I_0 \frac{2\nu_c}{\pi} N_0 \frac{m_0c^2}{2kT} \int_0^1 \frac{\exp -\left([m_0c^2/2kT]x\right) dx}{(\omega_0-\omega-\omega_0 x)^2 + 4\nu_c^2} \tag{3}$$

where I_0 is the total intensity emitted by a single electron and the resonant frequency is given by $\omega_0\,(1 - x)$, where x is v_p^2/c^2. For $\nu_c^2/\omega_0 \ll 2kT/m_0c^2$ and $\omega > \omega_0$ this integral is essentially zero for plasmas of laboratory dimensions; but if $\omega < \omega_0$ the integral is $(\pi/2\nu_c)\, e^{-(m_0c^2/2kT)}\,(1 - \omega/\omega_0)$ and Eq. 3 becomes

$$I(\omega) \sim I_0 N_0 \frac{m_0c^2}{2kT} \exp -\left[\frac{m_0c^2}{2kT}\left(1-\frac{\omega}{\omega_0}\right)\right] \tag{4}$$

That is, the emission at frequency ω is that due to the number of electrons radiating at a resonant frequency equal to ω. The emission and therefore the absorption is thus a very slowly varying function of the frequency compared to what it would be if there were no relativistic broadening. Since the absorption coefficient is a slowly varying function of frequency the Kramers-Kronig dispersion relation between the index of refraction and the absorption coefficient tells us that the index of refraction is very much closer to unity than it would be if the relativistic effects were not included. Thus, the change in refractive index at any point within the plasma is too gradual to result in a "shiny" highly reflective plasma. Therefore, we may compute the plasma emission by merely integrating Eq. 4 over the thickness of the plasma; if the result exceeds emission from a black body surface, however, the latter emission is predicted. As a result, we would expect fundamental cyclotron emission from a plasma with the brightness of a black body surface over a band width of

$$\omega_{0\,min} \Big/ \left[1 + 3(kT/m_0c^2)\log_e L\right] \lesssim \omega < \omega_{0\,max} \tag{5}$$

where $\omega_{0\,min}$ and $\omega_{0\,max}$ are given by the minimum and maximum field strengths respectively and L is the plasma length times a coefficient roughly equal to $(m_0c^2/2kT) \cdot (e^2N_0/m_0\omega^2)$, where e is the electronic charge.

So much for the fundamental emission. Schwinger[2] has evaluated the emission of harmonics of the fundamental cyclotron frequency of individual electrons as a function of electron velocity and angle, θ, of the emitted radiation to the plane perpendicular to the magnetic field. For $v^2 \ll c^2$ the emission of the harmonic frequency $r\omega_0 (r > 1)$ compared with the fundamental frequency $(r = 1)$ is given by

$$S_r(\theta) \sim \left[\{(\exp) V/2c\}\{\cos\theta\}\right]^{2(r-1)} \Big/ \sqrt{2r+1} \tag{6}$$

similar to the derivation of Eq. 4 we obtain

$$I(\omega) \sim I_0 N_0 \left(\frac{m_0c^2}{2kT}\right) \sum_r \frac{r \left[\{(r^2\omega_0^2 - \omega^2)/3\omega^2\}\{\cos^2\theta\}\right]^{r-1}}{\sqrt{2r+1}} \cdot \exp - \left[(m_0c^2/2kT)\left(\{r^2\omega_0^2 - \omega^2\}/3\omega^2\right)\right] \tag{7}$$

$$\lesssim I_0 N_0 \sqrt{\frac{\omega_0}{\omega}} \exp - \left[m_0c^2/2kT\right] \tag{8}$$

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2. J. S. Schwinger, Phys. Rev. 75 1912 (1949)

when integrated over θ and summed over r. Note that instead of the usual $I(\omega)d\omega$ we would have $I(\omega)d\omega/\omega$. Compared with the fundamental emission the harmonics are suffocated by the factor $\left[\left\{(r^2\omega_o^2 - \omega^2) / 3\omega^2\right\}\left\{\cos^2\theta\right\}\right]^{(r-1)}$

For plasmas of 10-100 cms in diameter the emission is an appreciable fraction of black-body emission for frequencies only a few multiples of the fundamental frequency. The emission decreases rapidly with increasing frequency. For plasma conditions $N_o = 10^{14}$, H = 5,000 gauss, kT = 50 kev, the emission for $\omega = 10\omega_o$ is less than 10^{-4} L of a black body where L is the plasma diameter. Since the emission depends only linearly on N_o while the power production in a Sherwood device depends quadratically on N_o the relative importance of the two processes depends linearly on N_o, higher densities decreasing the relative importance of the cyclotron emission. As Trubnikov and Kudryavtsev have observed only the high energy tail of the Maxwell-Boltzmann electron distribution is affected by energy losses due to cyclotron emission. The energy loss is further reduced by the reflectivity of the walls and field windings. The energy loss compared to a black body freely radiating in the absence of radiation reflectors is given by $A(1 - R) / (1 - R + RA)$ where A and R are the plasma absorptivity and wall reflectivity respectively. Thus compared to a black body the energy loss is $\sim 1 - R$ if $A > 1 - R$ and is $\sim A \; [1 - A/(1 - R)]$ if $A < 1 - R$.

** CHAIRMAN HARRIS: The floor is open for discussion.

DR. POST: I have a series of short remarks to make in response not only to the paper but to Jim Tuck's remarks, and may I treat them as a series of questions falling back from the approximation to the X. First let us suppose that you are wrong and the Russian calculation is totally right. The Russian calculation shows, as we all know, that the radiation in the fundamental cyclotron frequency is totally innocuous and it is the harmonics that are important. Furthermore, it simply points up, as we know, that the relativistic effects here are dominant; that it really illustrates it is the high energy electron irradiation that produces the majority of the radiation. So the question of whether these high energy electrons exist in the system and their rate of energy transport to them is important.

There is at least one case, and I have to cite our own. In the tensor mirror machine there are good reasons to believe that the high energy tail is, in fact, missing, the reason being that one cannot find electrons above a certain potential, which is the plasma potential, and these electrons are just the ones that we radiate. So if one puts any reasonable gas in the plasma potential he finds this effect is very small. Suppose I am wrong and the effect is large, is it, indeed, an effect that will lead to the net escape of energy. Here we can fall back on evidence from the theory of metals and the behavior of metals up to the short infrared, and find that any reasonable disposition of a surrounding conductance shell would reduce this radiation by at least two orders of magnitude, even if it existed.

Thirdly, I think there is real reason to suspicion (and there are several cases for this) that the electron temperatures in many of these devices may be a good deal lower than we have in the past assumed and the radiation for this reason alone becomes totally innocuous. So I quite agree with you that one can take one of the worst assumptions and it appears that the situation is very bad. However, I think there are many reasons for believing that these assumptions are not valid.

DR. BEARD: With reference to the conductance requirement, I don't know if you know I was at Livermore, spending the afternoon with Chuck Wharton. We explored this reflection business and what I have reported on is just the emission from the surface of the plasma. As mentioned in the report, emission from a plasma surrounded by highly reflecting walls is very much enhanced over the emission from a bare plasma surface.

DR. POST: I mean, you build up the energy of the fundamental density outside but you calculate the energy transported through that, and this is very low.

DR. ROSENBLUTH: I would like to make a comment. I have done essentially the same sort of calculation as Beard has, namely, to calculate the absorption of the plane wave in the plasma.

Now the situation is slightly more complicated because if you calculate the complete dispersion equation you find that there is a condition for the plasma radiation being the same as the single particle radiation. It is a very weak condition; namely, that the frequency of the emitted radiation that we are considering has been well above the plasma frequency, and this is the condition that is in general well satisfied. But if that condition is satisfied then the standard technics for calculating the absorption coefficient done the proper way give the exact same results as you get from the emission calculation by detailed balance.

I think there is one mistake that the Russians made; namely, they took only the propagation constant perpendicular to the magnetic field. In fact, the propagation coefficient is a strong function of the angle with the magnetic field; so that you really do not fill up the black-body distribution to the frequency which they mention but only in a narrow cone around the distribution. Numerical estimates would indicate that cases of interest may be a factor of 10 or 20 in radiation. I mean my feeling is that the Russian calculation is basically correct, although there is this factor of 10 or 20 down which is an important factor.

Then I furthermore agree with Dick in that I think essentially when you consider the effects of reflecting walls, the effect is by no means disastrous to thermonuclear machines.

DR. BERNSTEIN: I do a similar calculation and I agree substantially with Marshall. If you look at the coherent response of the plasma, the wave perpendicular to the magnetic field, you see that it is essentially transparent to radiation; so therefore any fluctuation to give radiation in this direction is a mistake and this substantiates the Russian claim. As Marshall contends, this serves only to cut the total emission down by a number, say, of no more than 20 or 30, and then there is conal emission.

RADIAL OSCILLATIONS OF CYLINDRICAL PLASMA CONFINED BY AXIAL MAGNETIC FIELDS

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Abstract

Radial oscillations of a cylindrical plasma, confined by an axial magnetic field have recently been observed (3), (4). In this paper these oscillations are discussed on the basis of the magneto-hydrodynamic equations. The effect of the proximity of a conducting wall and of differing mass distributions within the plasma are considered. It is found that the frequency is insensitive to these factors and depends only on the mass of plasma and the confining magnetic field. These oscillations should therefore provide a useful measure of the mass of gas swept up in a fast pinch device.

Introduction

In fast-pinch devices, whether produced by axial or azimuthal currents, the compressed plasma naturally does not come immediately to rest in its equilibrium position but undergoes a series of radial oscillations about a mean position which may itself be changing slowly with time. In the case of the z-pinch, in which the current is axial, these oscillations have been noted by Tuck (1).

Recently much interest has been shown in devices producing pinches by axial magnetic fields such as Scylla (2), Thetatron (3) and the apparatus used by Kolb (4). Niblett, using the Thetatron, and Kolb have observed radial oscillations in their experiments, and in view of

1. J. L. Tuck - Geneva II, p. 1860.
2. W.C. Elmore; E.M. Little; W.E. Quinn - Geneva II, p. 356.
3. G.B.F. Niblett - to be published.
4. A.C. Kolb - Geneva II, p. 345.

this it is interesting to discuss these oscillations in terms of conventional magneto-hydrodynamics. As a model for this system we consider a cylinder of perfectly conducting plasma of radius r confined within a concentric cylindrical conductor of radius R carrying an azimuthal current. The magnetic field is parallel to the axis. The type of oscillation in which we are interested is distinguished by the fact that it involves only radial motion. In the terminology of instability studies it corresponds to $m = o \quad k = o$.

We take the magnetic field to be purely axial and the motion purely radial, then the electrical field is azimuthal. Using the conventional equations of magneto-hydrodynamics the equation for small oscillations about a mean position of equilibrium can be derived along with appropriate boundary conditions. Analytic solutions can be obtained for certain idealised situations including the following:

Uniform Plasma

An elementary situation is that in which the density, pressure and field are uniform within the plasma, i.e. the plasma is confined by surface currents. In this case the angular frequency of oscillation can be expressed as

$$\omega = g(x) \sqrt{\frac{B^2}{4M}(1+\delta)}$$

where B is the confining magnetic field, M the mass of plasma per unit length and

$$\delta = \left(\frac{\gamma}{2} - 1\right)\frac{8\pi p}{B^2}$$

The quantity δ will generally be very much less than unity since for rapid radial motions the plasma has an effective γ near two, and the other factor is always less than unity.

The parameter x is connected with the pinch ratio $^{R}/r$ by

$$^{1}/x \doteq \frac{(1+\delta)}{2}\left(\frac{R^2}{r^2} - 1\right)$$

Values of g(x) are given in Table I.

1/x	0	.625	1.5	7.5	∞
g(x)	3.832	2.874	2.645	2.458	2.405

It will be seen that the maximum effect which the wall can have is to change the frequency by a factor 1.6. The influence of the gas pressure term, represented by δ, will be small, so that the frequency can, for practical purposes, be expressed in terms of the vacuum magnetic field, which is determined by the external current, and the mass of plasma per unit length of the discharge.

Non-uniform density.

Under the experimental conditions which are envisaged the plasma will initially have a greater density near its surface, having been swept up to some extent by the "snowplough" effect produced by the imploding current sheath and we would like to examine configurations which have a density profile which increases with radius. This can be done quite simply if we make the approximation that the adiabatic law with $\gamma = 2$, applies throughout the motion. [This approximation should be accurate as far as the oscillatory motion itself, for this involves two degrees of freedom; it will be less well satisfied for the equilibrium configuration.]

An analytic form of density profile which may represent the experimental situation fairly well is

$$\rho_o = \alpha\, r^{2s-2} \qquad s \geqslant 1.$$

then the angular frequency can be expressed

$$\omega = g_s(\Lambda)\sqrt{\frac{B^2}{4M}}$$

where $\Lambda = {}^R/r$ and g is given below

Λ =	1	1.25	1.5	2.0	4.0	∞
S = 1	3.83	3.16	2.87	2.64	2.46	2.40
2	4.44	3.17	2.77	2.49	2.28	2.22
4	5.56	3.22	2.73	2.40	2.18	2.12
∞	∞	3.33	2.68	2.31	2.07	2.00

The effect of varying the distribution of mass is very small indeed if the discharge is reasonably well compressed.

The limiting case $s = \infty$ corresponds to the mass being concentrated in a thin cylindrical shell, a distribution which can be treated as a problem in single particle dynamics, and for which

$$\omega = \frac{2\ \Lambda}{\sqrt{\Lambda^2 - 1}}\sqrt{\frac{B^2}{4M}}$$

Large Oscillations

Some guidance on the effect of finite amplitude can be gleaned from a study of the specially simple, but probably quite realistic, case of the plasma being confined in a thin shell.

In discussing large amplitude oscillation the mean position and the equilibrium no longer coincide so that equilibrium is not a convenient reference point. Instead it is convenient to use the maximum and minimum radii which the shell achieves in its oscillation denoted by r_2 and r_1 respectively.

The equation of motion for the shell is soluble in terms of elliptic integrals and the frequency of oscillation can be written

$$\omega = \frac{2\, h(x_1, x_2)}{\sqrt{1 - x_2^2}} \sqrt{\frac{B^2}{4M}}$$

where h is tabulated below and $x_i = r_i/R$.

x_2 \ x_1	0.2	0.4	0.6	0.8
0.2	1.00	-	-	-
0.4	1.033	1.0	-	-
0.6	1.099	1.067	1.0	-
0.8	1.223	1.196	1.137	1.0
1.0	1.571	1.571	1.571	1.571

It will be seen that provided the amplitude of oscillation is such that the maximum radius of the plasma is less than two-thirds that of the containing conductor the frequency of large oscillations is negligibly different from that of small oscillations.

Conclusions.

The frequency of oscillation of a cylindrical discharge confined by axial magnetic fields has been calculated in some idealised configurations. From the results one can deduce that for a reasonably well pinched discharge the frequency of oscillation is given by the characteristic frequency

$$\sqrt{\frac{B^2}{4M}}$$

multiplied by a factor which is insensitive to the ratio of plasma to magnetic pressure, to the actual distribution of plasma mass and to the amplitude of oscillation. We can conclude therefore that a measurement of this frequency allied to that of the confining field (which are about the simplest measurement one makes on a plasma device) form a good method of assessing the mass of gas which is swept up into the plasma and involved in the oscillation. It would, however, be very difficult to deduce temperature, the ratio of plasma to magnetic pressure, or the distribution of plasma density.

HYDROMAGNETIC ENERGY TRANSPORT IN IXION

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Abstract

The transport of energy by hydromagnetic waves is calculated, with view toward application to the heating process of the Los Alamos rotating plasma mirror device.

An understanding of the mechanism of conversion of ordered drift motion energy into random thermal energy is of great interest in analyzing the behavior of devices like Ixion, the rotating plasma machine at Los Alamos.[1] Here magnetic probe studies of the diamagnetic response seem to show that the original azimuthal drift mode of motion is transformed into a state which consists of a shell of thermal or turbulent motion located roughly halfway between the centerline and the outer electrode. The entire sequence of events occupies a relatively long time (of the order of a hundred microseconds) and the relative drift velocity of ions and electrons in the initial state is small. A plausible agency for effecting the energy transfer would be hydromagnetic waves, experimentally known to be excited in similar geometries. Mechanisms for the resonant damping of such waves, such as T. Stix's ion cyclotron heating process,[2] are likewise known and might account for the appearance of the observed final state.

As a first step in determining whether such a picture makes sense, the generation, structure, and energy transport of the appropriate hydromagnetic modes was examined. Using reasonable boundary conditions, one can then obtain cylindrical standing waves corresponding to the Ixion geometry, with fairly well defined shells of high energy density. To avoid complication, the motion of the plasma associated with the wave was treated by means of linearized Boltzmann equations, leading to the hydrodynamic approximation plus an equation of state, and the conductivity and viscosity of the fluid

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* Work performed under the auspices of the U. S. Atomic Energy Commission.

1. K. Boyer et al, Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, p/2383, <u>31</u>, 319 (1959).

2. T. H. Stix, Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy, p/361, <u>31</u>, 125 (1959).

were taken as empirical parameters. Dispersion relations for the three principal modes of propagation were obtained as well as their detailed polarization structure, and energy density distributions were calculated. The energy distributions for standing waves (corresponding to the modes commonly referred to as $\lambda = 2, 3$ magnetoacoustic modes) were found to be based on expressions in agreement with independent Russian results (see for example the work of Akhiezer and Sitenko[3]). The detailed results on the cyclotron heating rates in cylindrical shells and a comparison with experimental data will be presented in a subsequent paper.

3. A. J. Akhiezer and A. G. Sitenko, JETP 35, 116 (1958).

WAVES IN A PLASMA*

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Many papers that consider the effects of terminal motions, finite Larmor radius, collisions, and so forth on the propagation of plane waves through a plasma in the presence of a magnetic field have recently appeared.[1-4] The necessary mathematics obscures the origin of many of the predicted phenomena, and as these also depend critically on the range of frequency, plasma density, and the magnetic field that is considered, it has seemed worth while to view the complete range of these last three variables in the simple limit in which there are: (a) no density gradients; (b) no collisions; and (c) no thermal motions. The thermal motions affect mainly the slow waves whose phase velocity is comparable to the thermal motions. For this reason, among others, we shall be particularly interested to note the conditions under which slow waves exist.

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Under these restrictions, the mobility of an electron or ion in a magnetic field is a tensor quantity[5] that is particularly simple when it is expressed in components of the electric field which are either parallel (μ_p) or rotating about the magnetic field in a right-handed (μ_r) or left-handed (μ_ℓ) direction. In terms of the mobility tensor, we obtain the plasma conductivity,

$$\overleftrightarrow{\sigma} = \sum nq \, \overleftrightarrow{\mu}_{\pm} \tag{1}$$

by summing over the species of charged particles, and hence the effective dielectric coefficient

$$K = \begin{vmatrix} K_T & jK_H & 0 \\ -jK_H & K_T & 0 \\ 0 & 0 & K_p \end{vmatrix} = 1 + \quad j\omega \;_{o} \tag{2}$$

where

$$\begin{aligned} 2K_T &= K_r + K_\ell \\ 2K_H &= K_r - K_\ell \end{aligned} \tag{3}$$

The tensor (Eq. 2) is written in Cartesian, nonrotating coordinates. K_p and K_T are the components parallel and transverse to the magnetic field, and K_H is the component that gives the Hall effect. The last two components are given in terms of the rotating components by Eqs. 3.

For the particular case of a collisionless, cold, three-component (ions, electrons, and neutral molecules) gas the components of the dielectric tensor are

$$\begin{aligned} K_p &= 1 - \alpha^2 \\ K_r &= 1 - \alpha^2/(1 + \beta_+)(1 - \beta_-) \\ K \; &= 1 - \alpha^2/(1 - \beta_+)(1 + \beta_-) \end{aligned} \tag{4}$$

They are expressed simply in terms of the ratios

$$\alpha^2 = \frac{\omega_p^2}{\omega^2} = \frac{ne^2(m_+ + m_-)}{{}_o\, m_+ m_-\ \omega^2} \qquad (5)$$

$$\beta_{\pm} = \frac{\omega_{b\pm}}{\omega} = \frac{eB}{m_{\pm}\omega}$$

where α^2 is a measure of the plasma density n, $\beta_{\pm}$ of the applied magnetic field B, and α, β_+, β_- all vary inversely with the circular frequency ω of the electric field.

We now study plane waves by assuming that all quantities are proportional to

$$\exp j\omega(t - \vec{n} \cdot \vec{r}/c) \qquad (6)$$

where $\vec{n}$ is a vector normal to the wave whose magnitude n is the index of refraction for this direction of propagation. There should be no confusion in the use of the same letter in formula 5 because the plasma density will only appear implicitly in the symbol α. The phase velocity is

$$\vec{u} = \frac{c\vec{n}}{n^2} \qquad (7)$$

Substituting expression 6 in Maxwell's equations, we obtain

$$\vec{n} \times (\vec{n} \times \vec{E}) + \overleftrightarrow{K} \cdot \vec{E} = 0 \qquad (8)$$

This equation, among others, has been considered by Aström.[6] To obtain solutions, the determinant of its coefficients must vanish, and this gives the dispersion equation for the index of refraction n. This equation would, in general, be bi-cubic but, because the temperature has been neglected, the sixth degree terms cancel and we have the bi-quadratic equation

$$An^4 - Bn^2 + C = 0 \tag{9}$$

with

$$A = K_T \sin^2 \theta + K_p \cos^2 \theta$$

$$B = K_r K_\ell \sin^2 \theta + K_p K_T (1 + \cos^2 \theta)$$

$$C = K_p K_r K_\ell$$

whose discriminant is

$$D^2 = 4K_p^2 K_H^2 \sin^2 \theta + (K_r K_\ell - K_p K_T)^2 \cos^4 \theta \tag{10}$$

Here θ is the angle between the wave normal $\vec{n}$ and the applied magnetic field $\vec{B}$.

Because collisions have been neglected, the discriminant D^2 is always positive. Therefore n^2 is always real, and n either real or pure imaginary. This sharp distinction between conditions of propagation or attenuation exists in virtue of assumptions (a), (b), and (c).

The solutions of Eq. 9 are the indices of refraction

$$n = \left(\frac{B + D}{2A}\right)^{1/2} \tag{11}$$

associated with the two polarizations, but it is easier to understand the solutions of Eq. 9 if it is solved for the direction of propagation, θ, in terms of the index n:

$$\tan^2 \theta = - \frac{K_p \left(n^2 - K_r\right)\left(n^2 - K_\ell\right)}{\left(n^2 - K_p\right)\left(K_T n^2 - K_r K_\ell\right)} \tag{12}$$

In this form it is clear that for propagation along the magnetic field (θ=0) there are two waves

$$\begin{aligned} n_r^2 &= K_r \\ n^2 &= K_\ell \end{aligned} \tag{13}$$

that may be either propagated or attenuated according to the signs of K_r and K_ℓ, and the subscripts on the dielectric components indicate that they are right and left circularly polarized.

Similarly, for propagation at right angles across the magnetic field ($\Theta = \pi/2$) there are two waves

$$\begin{aligned} n_o^2 &= K_p \\ n_x^2 &= K_r K_\ell / K_T \end{aligned} \tag{14}$$

of which the first is polarized with the electric field parallel to the applied magnetic field. We shall call this wave "ordinary" because it is not affected by the magnetic field. The second wave, which we shall call "extraordinary," is transverse to the magnetic field but not transverse to the direction of propagation. It is made up of electric vectors rotating right- and left-handed around B, describing an ellipse in a plane perpendicular to B which contains the direction of propagation. Thus

$$\frac{2}{n_x^2} = \frac{1}{n_r^2} + \frac{1}{n^2} \tag{15}$$

The extraordinary velocity is intermediate between the right- and left-handed velocities.

For intermediate directions ($0 < \Theta < \pi/2$) the index is intermediate between the "principal indexes" given by Eqs. 13 and 14. If we make a polar plot of the phase velocity $\vec{u}$, we obtain two surfaces, called "normal wave surfaces," like those shown in Figs. 1, 2, and 3. Since D is never zero, the two wave surfaces do not intersect.

In crystal optics the term "ordinary" is used for waves that obey Snell's law, that is, those for which the wave surface is spherical. In our case neither surface is

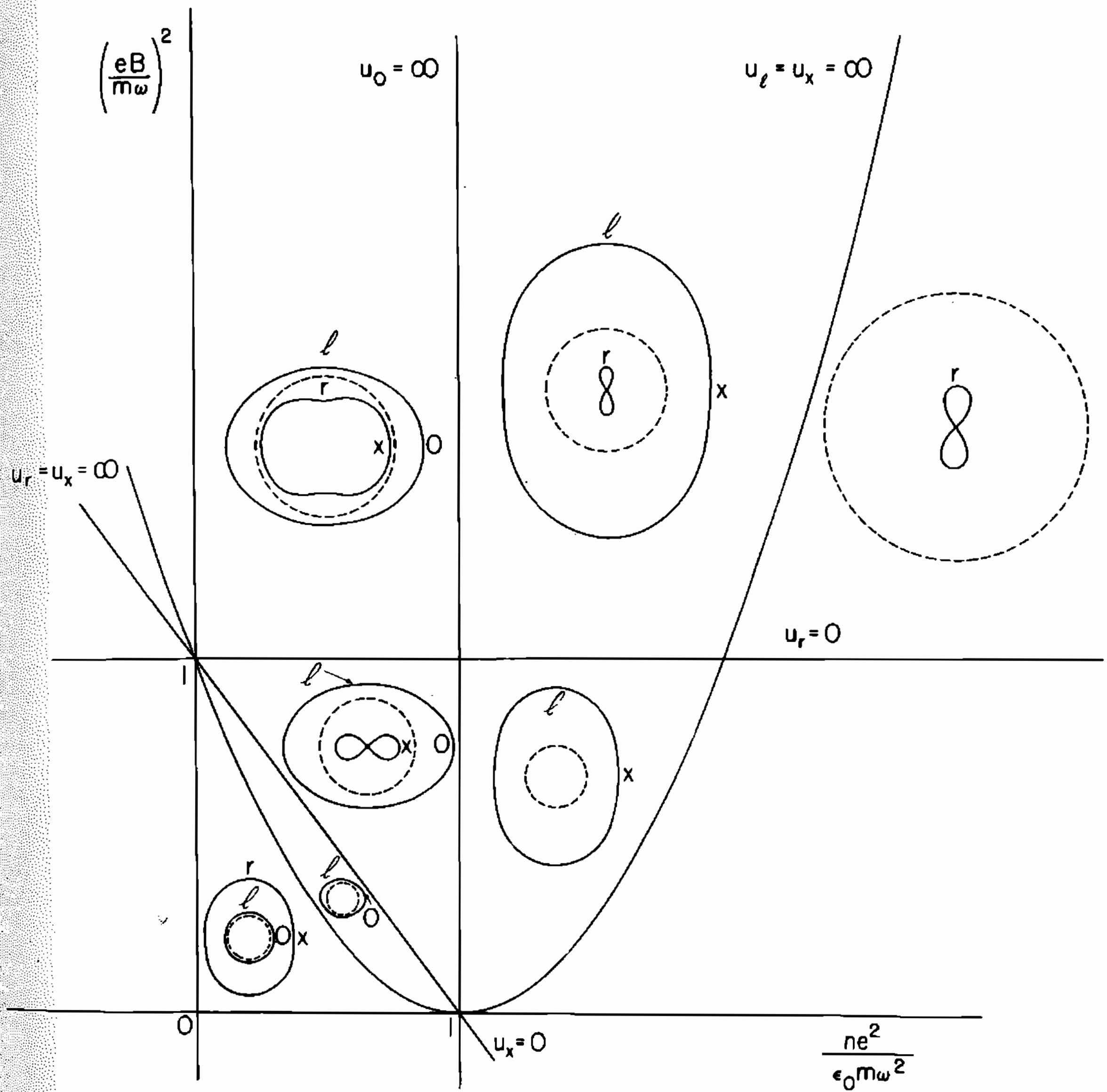

Fig. 1. Wave Normal Surfaces of a Plasma in a Magnetic Field (Effect of Electrons Only).

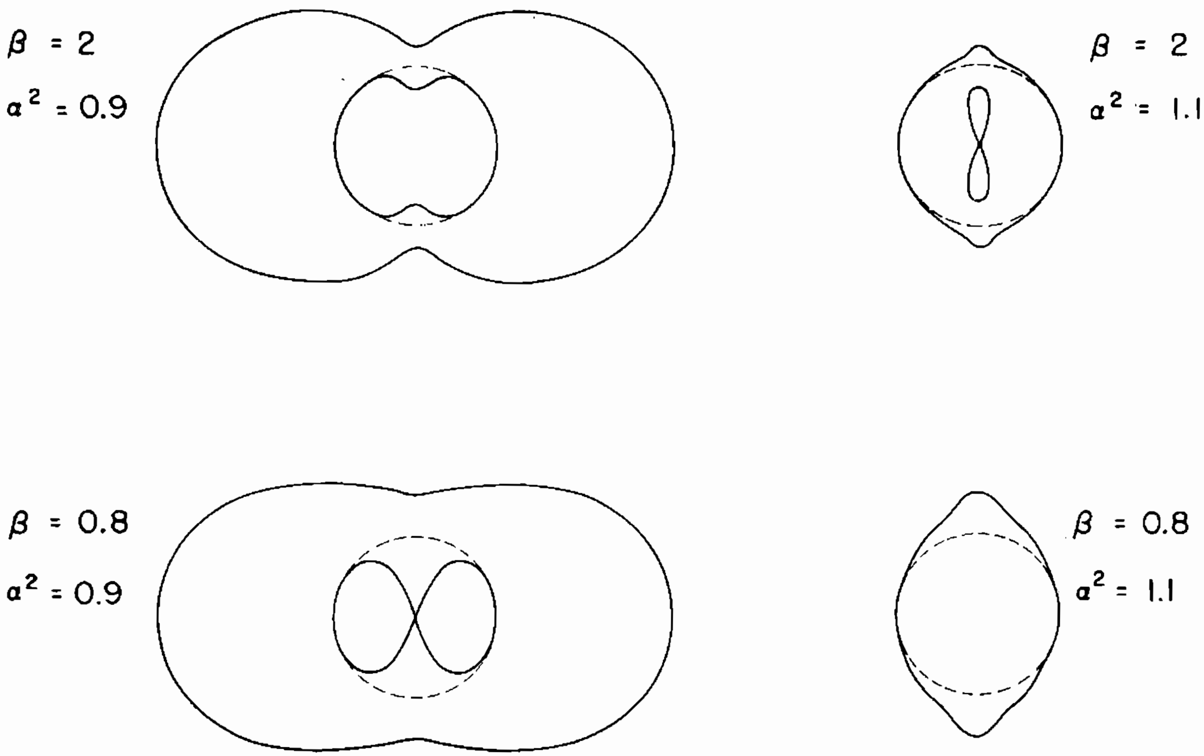

Fig. 2. Wave Normal Surfaces in the Vicinity of $\alpha^2 = 1$.

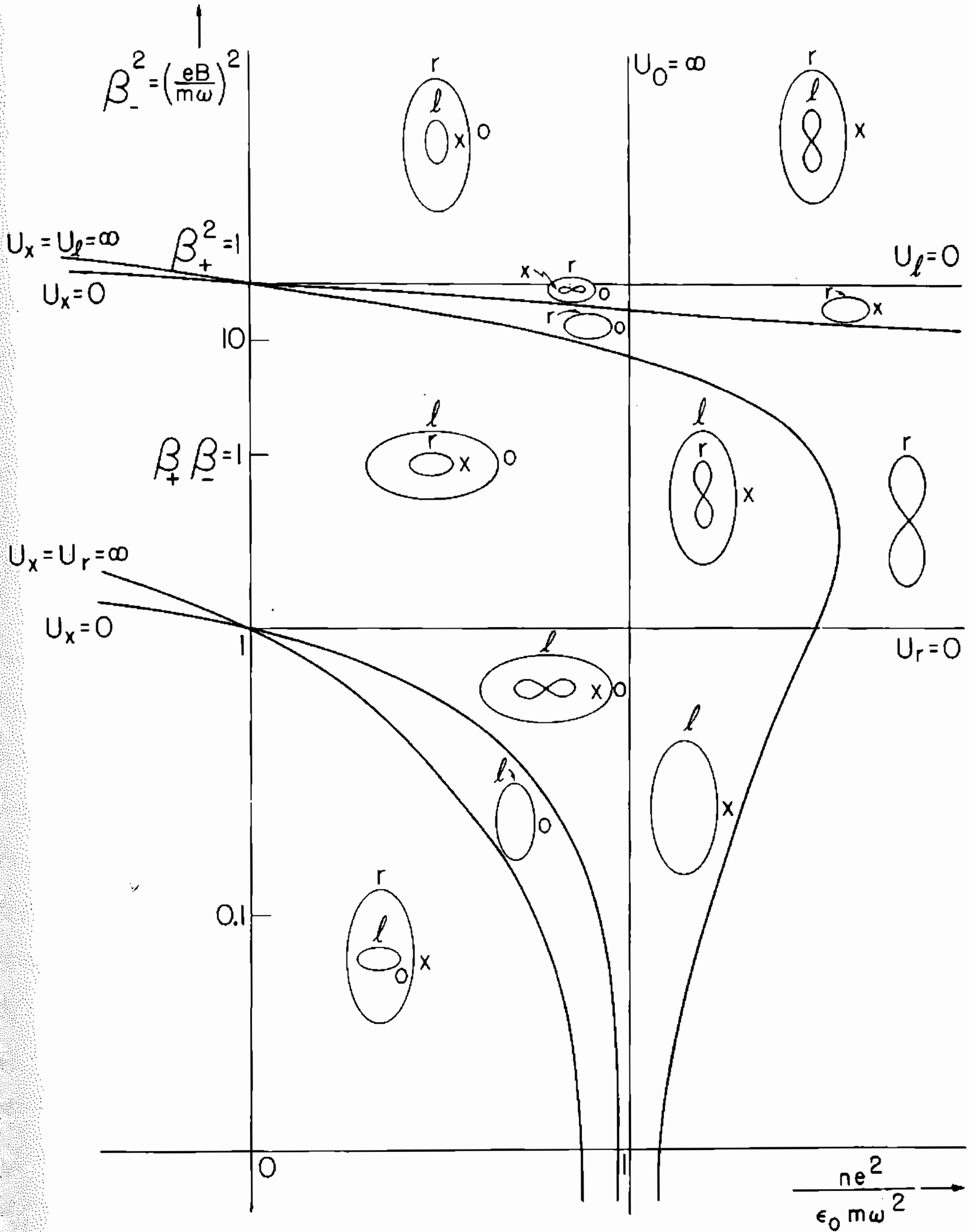

Fig. 3. Wave Normal Surfaces of a Plasma in a Magnetic Field, Including the Effects of Ions.

spherical except in limiting situations. The term "ordinary" does not apply to either complete wave surface; we use it in a different sense, and only for propagation normal to the magnetic field. (This is also Russian, but not Swedish, usage.) If we want a term for characterizing an entire wave surface, we should use "right-handed" and "left-handed," because this characterizes the direction of rotation of E around B for the entire surface. We must be cautious here, too, because the wave that we call "right-handed," rotates left-handed about the direction of propagation when it propagates along -B. A more satisfactory notation for an entire wave surface would be to denote it (rx), (ℓo), (x), and so forth.

We now wish to investigate the matter of which values of the parameters α, β_-, and θ give propagation ($n^2 > 0$), and which give attenuation ($n^2 < 0$). The boundaries of these regions are obviously the lines along which $n^2 = \infty$, $u = 0$, which we call "resonances," and those along which $n^2 = 0$, $u = \infty$, which we call "cutoffs."

The principal resonances are given by

$$K_r = \infty, \quad \beta_- = 1, \quad \text{Electron cyclotron resonance} \tag{16}$$

$$K_\ell = \infty, \quad \beta_+ = 1, \quad \text{Ion cyclotron resonance} \tag{17}$$

$$K_T = 0, \quad \alpha^2 = \frac{\left(1 - \beta_+^2\right)\left(1 - \beta_-^2\right)}{1 - \beta_+\beta_-}, \quad \text{Plasma resonance} \tag{18}$$

The first two justify our definition of "resonance." The third is an extension of the conventional use of "plasma resonance" which applies when there is a magnetic field, but note that "plasma resonance" does not occur at the "plasma frequency" ω_p. For high frequencies ($\beta_+ << 1$) plasma resonance occurs at

$$\omega^2 = \omega_p^2 + \omega_{b-}^2 \tag{18a}$$

and is represented on a plot of β_-^2 against α^2 (Fig. 1) by a diagonal straight line. In general, it is represented by a hyperbola

$$\omega^4 - \left(\omega_p^2 + \omega_{b-}^2 + \omega_{b+}^2\right) \omega^2 + \left(\omega_p^2 + \omega_{b+}\omega_{b-}\right) \omega_{b+}\omega_{b-} = 0 \tag{18b}$$

one branch of which goes through the points ($\omega_{b-} = 0$, $\omega = \omega_p$), and ($\omega_p = 0$, $\omega = \omega_{b-}$) and the other branch through ($\omega_p = 0$, $\omega = \omega_{b+}$) and ($\omega_p = \infty$, $\omega^2 = \omega_{b+}\omega_{b-}$). This last resonance, which occurs for large plasma densities ($n(M + m)c^2 >> B \cdot H$), is sometimes called the "hybrid cyclotron resonance" but its relation to the cyclotron frequencies is accidental. At large plasma densities the electrons and ions must move together in the direction of the wave normal, otherwise charge separation would occur, but this is prevented by Coulomb forces; however, they may move parallel to the wave surface. At the particular frequency $\omega^2 = \omega_{b+}\omega_{b-}$ the equations of motion[7] show that the electron and ion displacements in the direction of E are identical

$$\vec{\gamma}_+ \cdot \vec{E} = eE^2/(m-M) \tag{19}$$

whereas at right angles to E the electrons have large displacements. The resonance occurs because Coulomb and electromagnetic forces independently make the electrons and ions move together along E.

There is no resonance for the ordinary wave but there are resonances along directions other than the principal directions. These are found by setting $n = \infty$ in Eq. 12:

$$\tan^2 \theta_{res} = -K_p/K_T \tag{20}$$

and occur whenever K_p and K_T have different signs. For a given plasma these directions occur on a cone whose axis is along B and whose angle θ_{res} depends on the frequency. In directions near the resonant cone the phase velocity is slow and hence Čerenkov radiation is possible. Any electron in the plasma can have a "bow wave" which will be near the resonant cone (Fig. 4).

The resonant directions are also the directions in which plasma oscillations may occur, since it can be seen from Astrom's expressions for the components of the electric vector that this vector becomes normal to the wave surface at any resonance.

The principal cutoffs are given by

$$\left.\begin{matrix} K_r = 0 \\ K_\ell = 0 \end{matrix}\right\} \alpha^4 = \left(1 - \beta_+^2\right)\left(1 - \beta_-^2\right), \quad \text{Cyclotron cutoffs} \tag{21}$$

$$K_p = 0, \quad \alpha^2 = 1, \quad \text{Plasma cutoff} \tag{22}$$

The two cyclotron cutoffs form a continuous curve which is a parabola on the $\beta^2 - \alpha^2$ plot. The ordinary wave cuts off at the plasma frequency. There is no cutoff for the extraordinary wave. Neither are there cutoffs in other than the principal directions, because setting $n^2 = 0$ in Eq. 12 yields $\tan^2 \theta = -1$.

We are now ready to make a map of all possible wave surfaces by plotting β^2 against α^2 for all the principal resonances (Eqs. 16, 17, and 18) and cutoffs (Eqs. 21 and 22). Increasing the magnetic field would produce upward motion; increasing the plasma density would produce motion

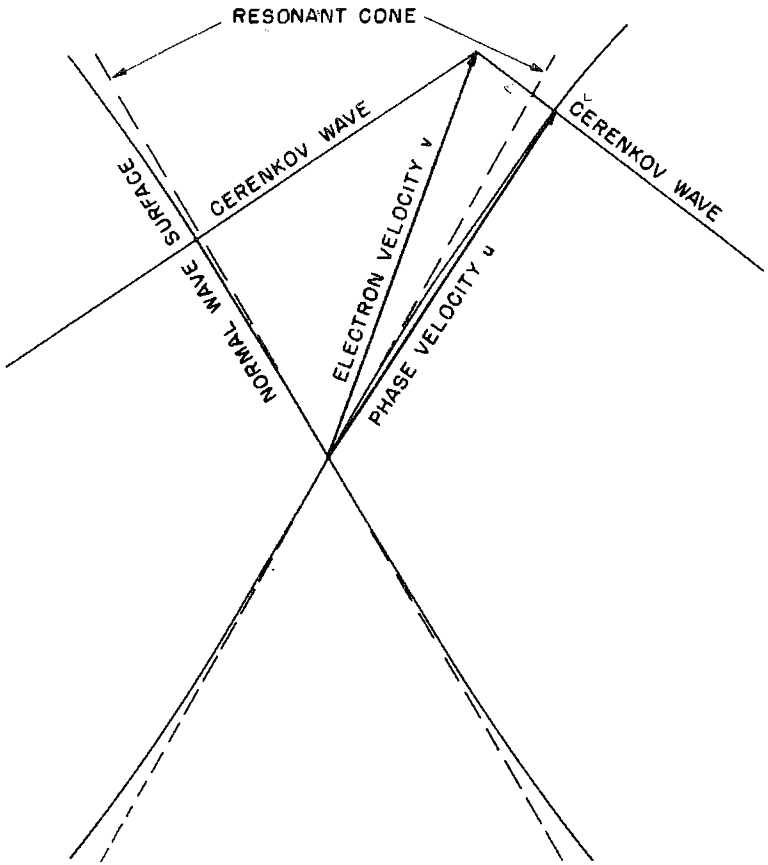

Fig. 4. Cerenkov Radiation Near the Resonant Cone.

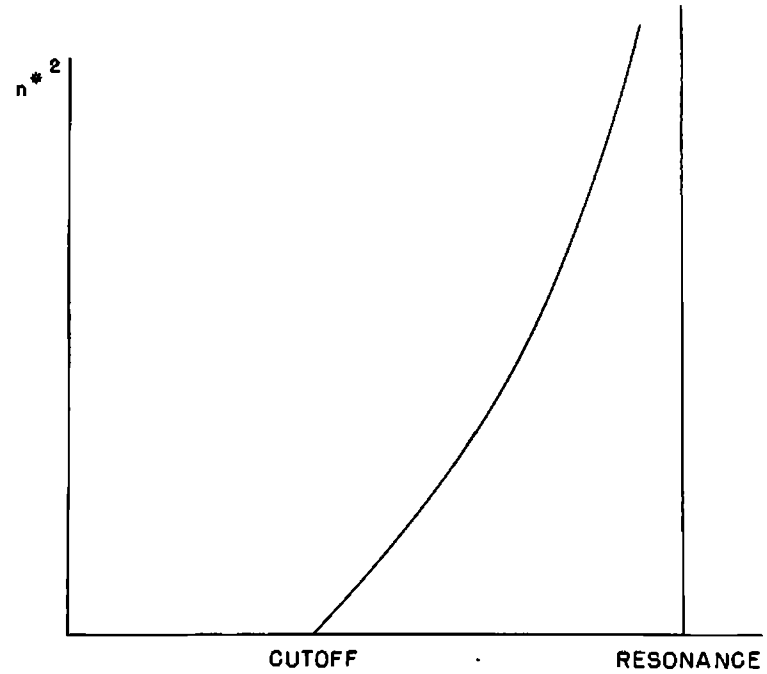

Fig. 5. Variation of Attenuation Between Cutoff and Resonance.

to the right; and decreasing the frequency for a given plasma and field would produce radial motion from the origin. Figure 1 shows only the high-frequency range ($\beta_+ << 1$), in which only the electrons can follow the oscillations. As a resonance or cutoff line is crossed, one or two of the waves in the principal directions disappears, or reappears, and hence the shape of the wave surface changes radically. Within each of the eight areas in which the plane is divided we have plotted the corresponding normal wave surface with the direction of B parallel to the β-axis, calculated for some specific point in the area. The free-space light velocity is given by the dotted circles as a reference. There is one area in which there is no wave surface, as all waves are attenuated in this area. In the remaining seven areas there is propagation in some directions, but in only two of them do the two waves exist for all directions. Thus a plasma is largely opaque or largely transparent according to the way in which you look at it. Three of the areas have figure-eight, or figure-infinity, wave surfaces. These are the areas where K_p/K_T is negative and there is a resonant cone. The two points at ($\beta = 0$, $\alpha = 1$) and ($\alpha = 0$, $\beta_- = 1$) are extremely singular because both resonance and cutoff lines intersect there. Only the presence of a magnetic field removes the confusion about whether the plasma frequency ω_p is a resonance or a cutoff. The ordinary cutoff line at $\alpha = 1$ is itself quite singular because on the low-density side of this cutoff the left-handed and ordinary waves are on the same wave surface, but on the high-density side it is the extraordinary wave that connects with the left-handed one. The transition is shown in Fig. 2, in which four wave surfaces close to the plasma frequency are shown. At the plasma frequency the wave surfaces consist of a sphere close to the velocity of light

$$\frac{1}{n^2} = \frac{1}{n_x^2} = \frac{1 - \beta_+}{\beta_- \beta_+^2 - 1} \qquad (23a)$$

but the polar points on this sphere are missing. They are replaced by two external points

$$\frac{1}{n^2} = \frac{1}{n^2} = 1 + \frac{1}{\beta_-} \qquad (23b)$$

and two internal points (if they are real)

$$\frac{1}{n^2} = \frac{1}{n_r^2} = 1 - \frac{1}{\beta_-} \qquad (23c)$$

On the left of $\alpha^2 = 1$ the sphere has dimples that connect with the internal points (or the origin). On the right of $\alpha^2 = 1$ the sphere has projections that connect with the external points.

As we approach the line $\alpha = 1$ from the right above cyclotron resonance, or from the left below cyclotron resonance, the resonance cone becomes very narrow. Thus although $\alpha = 1$ is not a resonance, it is always very close to a resonance for propagation very nearly along B.

The entire range of frequencies is shown in Fig. 3, but logarithmic scales have had to be used and this obscures the simple shape of the resonance and cutoff lines. Even so, a small mass ratio of 4 had to be chosen so that the two small areas near ion cyclotron resonance would remain visible. There are now 13 areas with 12 distinct wave surfaces.

In the limit of low frequencies the figure-eight in the upper right-hand corner becomes two spheres tangent at the origin, and the elliptical figure becomes a sphere tangent externally to the two previous spheres. This sphere obeys Snell's law and is called the "ordinary wave" by Aström. We denote it (rx), and we have

$$\frac{1}{u_{rx}^2} = \frac{1}{c^2} + \frac{n(M+m)}{B \cdot H} \tag{24a}$$

where $(B \cdot H/n(M+m))^{1/2}$ is the Alfvén velocity. The other, termed "extraordinary wave" by Aström, is given by

$$u_{\ell}^2 = u_{rx}^2 \cos^2 \Theta \tag{24b}$$

In regions where n^2 is negative the exponential (B) may be written

$$\exp\left(j\omega t - 2\pi \vec{n}^* \cdot \vec{r}/\lambda_o\right) \tag{25}$$

where λ_o is the free-space wavelength and $2\pi n^* = 2\pi jn$ is the attenuation per free-space wavelength. This attenuation is nowhere shown on our diagrams, but it is evident that n^{*2} rises linearly beyond any cutoff and jumps from zero to infinity at any resonance (Fig. 5). Because we have removed all absorption mechanisms from our theory, a semi-infinite plasma will be perfectly transparent when $n^2 > 0$, and perfectly reflecting when $n^2 < 0$. A slab of plasma whose thickness is a finite number of free-space wavelengths will still be perfectly reflecting near a resonance, but near a cutoof considerable radiation may be transmitted.

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6. E. Astrom, Arkiv Fysik 2, 443 (1950).
7. W. P. Allis. op. cit., p. 4.

OSCILLATIONS OF A FINITE COLD PLASMA IN A STRONG MAGNETIC FIELD

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Project Matterhorn, Princeton University

Most treatments of plasma oscillations have been given for plasmas of infinite extent. Such treatments give information on the propagation of electromagnetic waves inside a plasma, but give no indication of the coupling between these waves and the electromagnetic fields outside the plasma. This coupling determines the radiation from, say plasma oscillations, as well as the response of the plasma to externally applied fields, where the fields may be either wave fields or near fields produced by currents and charges near the plasma surface. Since the electromagnetic field affords one of the most fruitful means for investigating the behavior of plasmas, it is important to know the size and effects of this coupling.

We have considered the oscillations of a bounded plasma situated in a strong magnetic field. In order to facilitate the treatment of the problem, we make the following assumptions:

a) The magnetic field is of such strength that motions perpendicular to it are negligible;

b) Thermal motions of the electrons are negligible;

c) The electrons behave like a charged continuous fluid;

d) The ions constitute a uniform, fixed neutralizing background;

e) The amplitude of oscillations is so small that the linearized equations of motion are applicable;

f) The mass motion of the electrons in the unperturbed plasma vanishes.

If the constraining magnetic field is taken in the z-direction, the linearized equations of motion for the perturbation quantities inside the plasma are shown in Slide 1. The plasma geometries we have considered are the infinite slab and cylinder with the constraining field B_o parallel to the surface.

As for boundary conditions at the plasma surface, the large zero order magnetic field constrains electrons to move parallel to the surface and hence no surface charge can accumulate and hence all components of E are continuous across the surface. Since the plasma density is everywhere finite, there are no sheet currents at the plasma surface. Thus all components of B are continuous.

$$\vec{e}_z m \frac{\partial V_z}{\partial t} = -\vec{e}_z e E_z ,$$

$$\frac{\partial N}{\partial t} + n \frac{\partial V_z}{\partial z} = 0 ,$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} - \vec{e}_z \frac{4\pi e n}{c} V_z$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} ,$$

$$\nabla \cdot \vec{E} = -4\pi e N ,$$

$$\nabla \cdot \vec{B} = 0 .$$

Slide 1.

In order to completely specify the problem, boundary conditions at large distances must be given for the fields such as periodic or reflecting ones (which conserve energy). However, in the treatment of the problem of the radiation due to plasma oscillation it is convenient to abandon these energy conserving boundary conditions at large distances and admit the presence of perfectly absorbing boundaries.

Before proceeding on to special problems, it is worthwhile to exhibit an orthogonality relation satisfied by the normal modes. It is not necessary to know the complete structure of these modes but only that the field quantities are of the form $A(x, y, z, t) = A(x, y) \exp i(\omega t + k_z z)$.

It is easy to show from the basic set of equations that all field quantities are determined when E_z is determined. If $E_{z1}(x, y)$ and $E_{z2}(x, y)$ are two normal modes of the system (with the same k_z) and ω_1 and ω_2 are the corresponding normal frequencies, then we find the orthogonality relation shown in Slide 2.

$$\frac{1}{c^2}(\omega_1^2 - \omega_2^2) \left[\int_{U^o+U^i} E_{z1}E_{z2}\, dx\, dy - \frac{\nu^2\Omega^2}{\omega_1^2}\frac{c^2}{\omega_2^2}\int_{U^i} E_{z1}E_{z2}\, dx\, dy\right] = 0 .$$

$$\frac{1}{c^2}(\omega_1^2 - \omega_2^2) \left[\int_{U^o+U^i} E_{z1}E_{z2}\, dx\, dy + \frac{c^2\Omega^2 m^2}{n^2 e^2}\int_{U^i} N_1 N_2\, dx\, dy\right] = 0 .$$

Slide 2.

Here U^o and U^i are the regions exterior and interior to the plasma in a constant z plane and Ω^2 is the square of the plasma frequency. This equation can alternatively be written as shown in Slide 3 which exhibits explicitly the particle contribution to the modes. These modes can be normalized so that the term in brackets is δ_{12} if the modes are discrete or $\delta\,(1-2)$ if they are members of the continuum.

$$k_x^{o2} + k_y^2 + k_z^2 = \omega^2/c^2$$

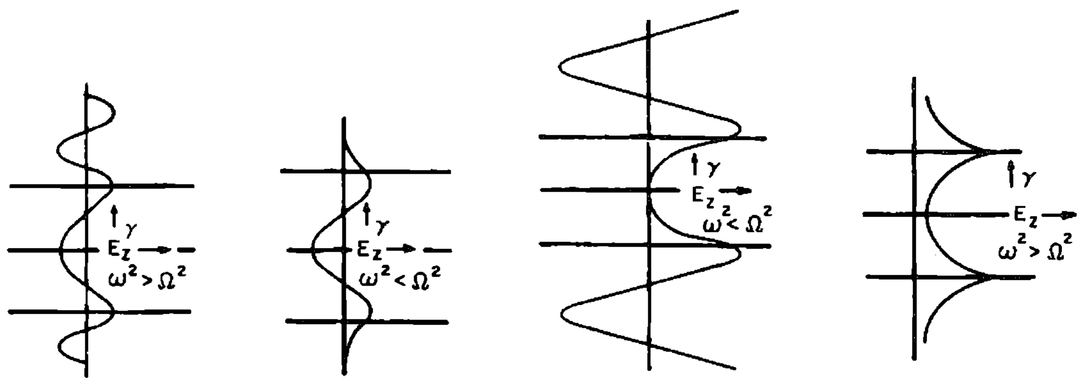
$$k_x^{i2} = k_x^{o2}\,(1 - \Omega^2/\omega^2) - \mu^2\Omega^2/\omega^2$$

$$\Omega^2 = 4\pi m e^2/m$$

Slide 3.

If we continue the Fourier decomposition of the modes in x and y say for the slab situation, we find the following dispersion relation relating the wave numbers inside and outside the plasma to the frequency.

Because of the symmetry of the situation, the modes fall into either even or odd forms in x and their general character is exhibited in Slide 4. Case A is the situation obtained when $(k^o{}_x)^2 > 0$, $(k^i{}_x)^2 > 0$ and represents waves propagating in the x-direction both inside and outside the plasma. This is a situation arrived at when the frequency $\omega^2 > \Omega^2$, and where the phase velocity along the surface exceeds the velocity of light c. (For ease of presentation, we take $k_y = 0$ unless otherwise indicated.)



SLIDE 4

Case B where $k^i{}_x{}^2 > 0$, $k^o{}_x{}^2 < 0$ are the modes which propagate in the x-direction inside the plasma, but not outside. These are waves trapped in the plasma. This situation obtains when $\omega^2 < \Omega^2$ and the phase velocity along the surface is less than c. In this case, a period equation shown in Slide 5 must be satisfied in virtue of the boundary conditions at the plasma surface and this in turn restricts $k_x{}^o$, $k_x{}^i$ and ω to discrete values for a given k_y, k_z. In this slide the period equation is given as well as the values of $k^o{}_x$, $k^i{}_x$, ω for k_z given for frequencies near the plasma frequency.

If we refer to Slide 4, we see Case C which propagates on the outside but not on the inside. These are waves excluded from the plasma.

Modes of type D are not allowed in the present problem because of the break in the derivative. Conditions other than those here realized (surface charge) can give rise to modes of this type.

I would like now to touch briefly on several problems which can be treated using the previous normal mode analysis.

For Case B.

$$\left\{\begin{matrix} \tan \\ -\cot \end{matrix}\right\} (k_x^i \delta) = |k_x^o|/k_x^i \quad \left\{\begin{matrix} \text{even} \\ \text{odd} \end{matrix}\right\}$$

If $\mu = 0$, $\omega^2 \simeq \Omega^2$ then with $\epsilon = \Omega - \omega$

we find

$$k_x^{o2} = (\Omega^2/c^2 - k_z^2) + 2\Omega\,\epsilon/c^2$$

$$k_x^{i2} = 2(\epsilon/\Omega)\,(k_z^2 - \Omega^2/c^2)$$

Then $\epsilon/\Omega \simeq$

$$\epsilon/\Omega \simeq \left\{\begin{matrix} (n + 1/2)^2 \\ n^2 \end{matrix}\right\} \pi^2/2\,(k\delta + 1)^2 \left\{\begin{matrix} \text{even} \\ \text{odd} \end{matrix}\right\}$$

Slide 5.

The first one that I shall just mention is the scattering of a plane e.m. wave by a plasma cylinder. The previous normal mode analysis for slab situation was repeated for the cylindrical situation with essentially similar results. These modes were superposed so as to add up in the usual way to an incoming plane wave of unit amplitude plus an outgoing cylindrical wave. The amplitudes and phase shifts of the partial waves are derived as well as the differential and total cross section. These results are of course functions of the plasma frequency and radius of the cylinder and may be of use in plasma diagnostics.

A second similar problem is that of reflection and transmission of radiation for a plasma slab. That is, we consider the problem of a plane wave with certain wave vector incident upon the slab and investigate the amplitudes of the transmitted and reflected waves. Again supposing the normal modes so as to yield only an outgoing wave on the side opposite to incidence, gives the reflected and transmitted amplitudes, and thus intensities. Slide 6 shows the transmission coefficients ($k_y = 0$) for Cases A and C, with their dependence on the frequency, plasma frequency, and thickness. The usual interference properties of thin films are here realized. (For instance, there is total transmission whenever $\lambda^i{}_\delta = n\pi/2$.)

Another interesting problem which we considered was the formulation of the response of the plasma to any arbitrary distribution of sources on planes on either side of plasma slab. We carried out in detail the

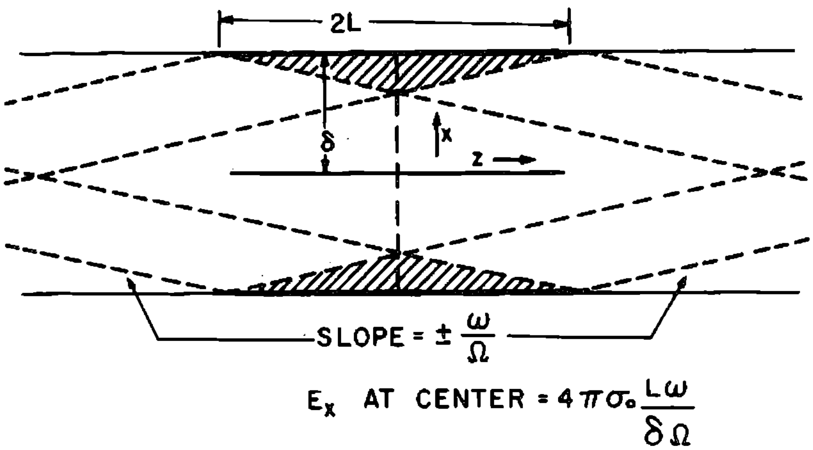
Case A.

$$T = \left[1 + \frac{\Omega^4}{4\omega^4}\,(1 - \Omega^2/\omega^2)^{-1}\,\sin^2(2k_x^i\delta)\right]^{-1}$$

Case C.

$$T = \left[1 + \frac{\Omega^4}{4\omega^4}\,(\Omega^2/\omega^2 - 1)^{-1}\,\sin h^2\,(2|k_x^i|\delta)\right]^{-1}$$

Slide 6.

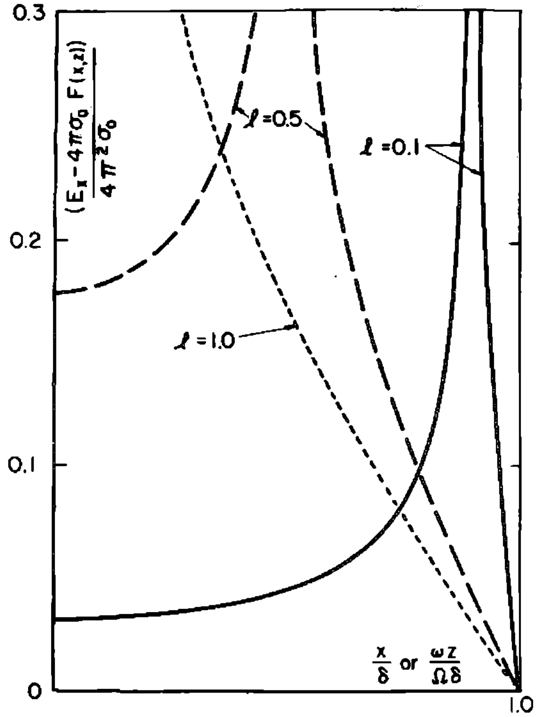


SLIDE 7

problem of oppositely charge finite condenser plates placed close to the plasma surface driven at a frequency $\omega << \Omega$. Slide 7 shows schematically the geometry and standing wave pattern of the steady state behavior of the field which has strong maxima along the dotted lines. The shaded area shows the region of penetration of the vacuum field $4\pi\sigma_0 e^{i\omega t}$. The possibility of getting strong low frequency fields into the plasma is an interesting one. Slide 8 shows a plot of E_x at $z = 0$ as a function of x for various values of ℓ , where $\ell = L\omega/\delta\Omega$.

The last problem I wish to discuss is the radiation emitted by plasma oscillations of the slab. There are several approaches to the problem. We have formulated the initial value problem for the plasma in terms of normal modes so that we can pluck the plasma and watch the development of the radiation field in time. We have also formulated the problem of the response of the plasma in the presence of externally driven currents and charges and mechanical forces. Again, under these circumstances we can sit back and watch the evolution and steady state behavior of the field. A third method which yields results in close accord with the other two is to merely alter the boundary condition at large distance so as to absorb all incident radiation.

The analysis now is formally similar to the other normal mode cases except now ω is complex with positive imaginary part to conform



E_x vs $\frac{x}{\delta}$ or $\frac{\omega z}{\Omega\delta}$

SLIDE 8.

$$\begin{Bmatrix} \tan \\ -\cot \end{Bmatrix} (k_x^i\delta) = ik_x^o k_x^i \quad \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$$

Let $\omega = \Omega + \epsilon$

then $\epsilon/\Omega = \begin{Bmatrix} (n + 1/2)^2 \\ n^2 \end{Bmatrix} \dfrac{\pi^2}{2(k^2\delta^2 + 1)^2} (k^2\delta^2 - 1 + 2ik\delta) \begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$

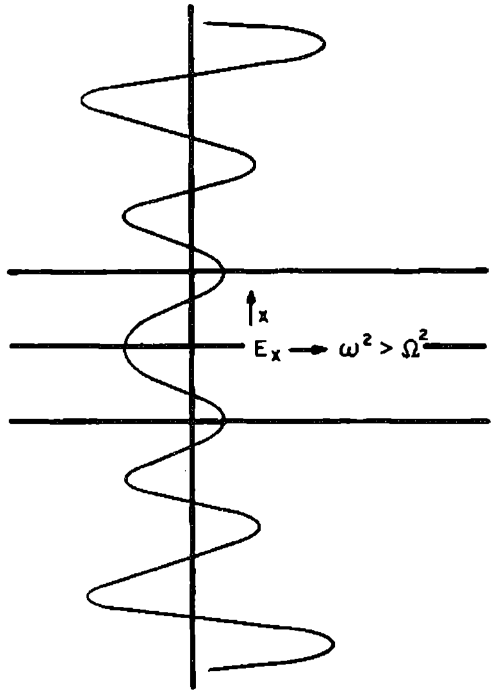
where $k = (\Omega^2/c^2 - \nu^2)^{1/2}$.

Also

$$k_x^o = k + \frac{\Omega^2}{kc^2}\frac{\epsilon}{\Omega} .$$

Slide 9.

with the absorbing boundary condition. This in turn demands k_x^o and k_x^i to be complex, and the real parts of k_x^o and ω must have the same sign (for $x > 0$) so as to yield only outgoing radiation. Slide 9 shows the period equation demanded by boundary conditions at the plasma surface. For the case $k_y = 0$ and ω close to Ω we show the characteristic frequencies and wave numbers and the explicit exponentiation in space and time. Slide 10 shows the form of these modes.



SLIDE 10

We are currently extending these investigations to include (a) the effects of a finite rather than infinite equilibrium magnetic field, (b) the ion dynamics, and (c) the effects of temperature.

SOME ADDITIONAL RESULTS ON WAVES IN A PLASMA IN A MAGNETIC FIELD

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The general dispersion relation has been computed which characterizes the small motions about a static equilibrium in a uniform external magnetic field of a fully ionized, relativistic plasma. It is assumed that collisions are negligible. Among the various results obtained are information on certain beam instabilities, hydromagnetic instabilities associated with anisotropies of the equilibrium distribution function, and the propagation of electromagnetic waves.

Of particular interest are the results on the propagation of waves in directions skew to a strong equilibrium magnetic field. When the velocity distribution of the electrons is that corresponding to thermal equilibrium, such waves whose frequencies are harmonics of the gyration frequency are little damped. Application of Kirchoff's law then indicates that the synchrotron radiation should be correspondingly small. This is in agreement with the results of Rosenbluth.

This work is being written up for publication.

C. THEORIES PERTINENT TO SPECIFIC EXPERIMENTS

THE INTEGRAL INVARIANT FOR ADIABATIC PARTICLE MOTION

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Abstract

The usual first-order particle drifts conserve the integral invariant, $I = \int u_{\parallel}\, dl$. A proof will be outlined and applications to particle motions in magnetic fields will be given. Further details may be found in UCRL-5615, "Stability of the Adiabatic Motion of Charged Particles in the Earth's Field," by T. G. Northrop and E. Teller. A paper has also been submitted to the PHYSICAL REVIEW.

CRITICAL CURRENT FOR BURNOUT IN AN OGRA-TYPE DEVICE

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Abstract

A complete algebraic analysis has been obtained for the variation of the steady state ion density n_+ with injected current I in an OGRA-type fusion device (i.e., a device based on trapping of ions by breakup of energetic molecular ions on collision with either the background gas or trapped ions). The most general variation of n_+ with I is shown to be an s-curve with at most three roots of n_+ for a given input I. A physical interpretation of these three roots is given. In addition algebraic expressions are obtained for the two currents at which the bends in the s-curve occur. It will be necessary to attain the larger current in order to build up a high density plasma when the density is being increased from below. On the other hand, once the high density has been achieved it may be maintained by steady injection of a current larger than the lower value.

In two previous publications,[1-2] an expression was derived for the critical current at which formation of a plasma by high-energy injection will begin. This previous expression was a case in which the trapping mechanism (although not specified in detail) was localized and did not depend on either the neutral gas in the device, the trapped ion density or the dimensions of the system. A trapping mechanism of this sort is provided by the arc in DCX.[3]

The situation is quite different in a proposed fusion device such as OGRA.[4] Here the injected molecular ions have a long mean-free-path L before they strike the injector snout and trapping occurs by virtue of the dissociation of the molecule on collision with either the background gas (cross section $\sigma_B{}^0$), the trapped ions ($\sigma_B{}^+$), or other molecular ions in transit ($\sigma_B{}^{2+}$). Nevertheless, one might suspect on physical grounds that a critical current also exists in this case and indeed such an expression has been found. The result is somewhat more complex than in the case of DCX because of a feedback which is inherent in the gas-breakup scheme. The onset of neutral burnout results in a reduction of

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2. A. Simon, The Phys. of Fluids, (in press).
3. C. F. Barnett et al., Proc. Second Geneva Conf. 31, 298 (1958).
4. I. V. Kurchatov, J. Nuc. Energy 8, 168 (1958).

the neutral breakup centers as well as an increase of the ion breakup centers and hence has a back effect on the input trapped current.

A complete algebraic analysis of the steady state equations has been achieved in the case when one can neglect the contribution of the molecular ions to burnout or to breakup of other molecules as compared to the effect of the trapped ions and the neutral gas. (This is a highly valid approximation in almost all cases of interest.) The total mean free path λ of the injected molecular ions is then:

$$\frac{1}{\lambda} = \frac{1}{L} + \hat{N}_o \sigma_B^{\,o} + n_+ \sigma_B^{\,+} \tag{1}$$

where $\hat{N}_o$ is the average neutral density external to the plasma region and n_+ is the trapped energetic ion density. (It has been assumed that the slow ions resulting from ionization of the neutrals in the plasma region contribute equally to breakup as do the neutrals themselves. The sum of the slow ion and neutral densities in the plasma interior should remain equal to the external neutral density even after burnout.) The probability that a molecule will break up after a path length x is then

$$p(x) = e^{-x/\lambda} \frac{dx}{\lambda_B} \tag{2}$$

where

$$\frac{1}{\lambda_B} = \hat{N}_o \sigma_B^{\,o} + n_+ \sigma_B^{\,+} \tag{3}$$

Hence the percentage trapping is found by integrating Eq. (2) over all space. The result is

$$\%\ \mathrm{B.U.} = \frac{\lambda}{\lambda_B} = \frac{\hat{N}_o \sigma_B^{\,o} L + n_+ \sigma_B^{\,+} L}{1 + \hat{N}_o \sigma_B^{\,o} L + n_+ \sigma_B^{\,+} L} \tag{4}$$

This leads to the following steady state equation for the ion density:

$$\frac{I}{V} \; \frac{\hat{N}_o \sigma_B^{\,o} L + n_+ \sigma_B^{\,+} L}{1 + \hat{N}_o \sigma_B^{\,o} L + n_+ \sigma_B^{\,+} L} = \frac{\hat{N}_o n_+ \sigma_{cx} v}{1 + \frac{\bar{\ell} v}{v_o} n_+ \sigma_d^{\,+}} + n_+^2 \sigma_c v P \tag{5}$$

where

$$\hat{N}_o = \frac{\left(\frac{\Gamma}{\theta}\right) I + N_o}{1 + \left(\frac{\sigma}{\theta}\right) \frac{n_+ \sigma_d^{\,+} v V}{\left(1 + \frac{\bar{\ell} v}{v_o} n_+ \sigma_d^{\,+}\right)}} \tag{6}$$

Here I is the injected (number) current of molecular ions, V the plasma volume, $\bar{\ell}$ the mean chord length of the plasma volume, v the energetic ion velocity and v_o the neutral atom velocity. The charge exchange cross section is denoted by σ_{cx}, the sum of the charge exchange and ionization cross sections is denoted by σ_d^+ ($\sigma_d^+ = \sigma_i + \sigma_{cx}$), and σ_c is the "effective 90° coulomb scattering cross section." Finally, the pumping speed of the system is denoted by θ, N_o is the initial neutral density before the beam is turned on, Γ is the fraction of input molecular ions which come back as neutrals ($\Gamma \leq 2$). σ is the fraction of slow ions which do not return to the system as neutrals after striking the walls and P is the usual mirror loss probability.

Equations (5) and (6) combined constitute an implicit equation in the variables n_+ and I. Thus we have

$$f(n_+, I) = 0 \tag{7}$$

It can be shown that I is uniquely determined by a choice of n_+, and conversely that there are either three or one real positive values of n_+ for any given I. As a result, the variation of n_+ with I has the general form shown in Fig. 1,[5] whereas the corresponding curve for DCX has the form shown in Fig. 2.

The multiple roots occurring in Fig. 1 have a straightforward physical interpretation. In region 1 neutral burnout has set in. The steady state solution is achieved by balance between charge exchange loss of the trapped ions and feed by breakup of the molecular ions on the neutral background (mirror loss is negligible). The second solution in region 2 corresponds to the point at which the ion density has risen and the neutral density has fallen such that the breakup on the ions now is the same as the previous breakup on the neutral gas. The charge exchange loss remains the same since it is proportional to the product $n_o n_+$ (where n_o is the average neutral density in the plasma region) and since, after burnout, $n_o \sim 1/n_+$. The final root of region 3 corresponds to the point at which mirror loss becomes more important than charge exchange loss. It is clear that roots 1 and 3 are stable while root 2 will be unstable.

An implicit equation for the upper critical current (U.C.C. in Fig. 1) has been found and is as follows:

$$I_{U.C.C.} = \frac{\left(\frac{\Gamma}{\theta} I + N_o\right)\sigma_{cx} vV \left(\frac{\phi - 1}{\phi}\right)\left[1 + \Lambda\, \sigma_B{}^o L + \left(\frac{\phi - 1}{K}\right)\sigma_B{}^+ L\right]}{K\left[\Lambda\, \sigma_B{}^o L + \left(\frac{\phi - 1}{K}\right)\sigma_B{}^+ L\right]} \tag{8}$$

where

$$\phi = \left[\left(\frac{\Gamma}{\theta} I + N_o\right)\frac{\sigma_B{}^o\, \sigma}{\sigma_B{}^+\, \theta}\; \sigma_d{}^+\, vV\right]^{1/2} \tag{9}$$

$$K = \left(\frac{\bar{\ell}}{v_o} + \frac{\sigma}{\theta} V\right) v\sigma_d{}^+ \tag{10}$$

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5. A recent paper by I. N. Golovin (Harwell, April 1959, unpublished) states that Kuznetsov and coworkers have obtained numerical results indicating a behavior of this sort.

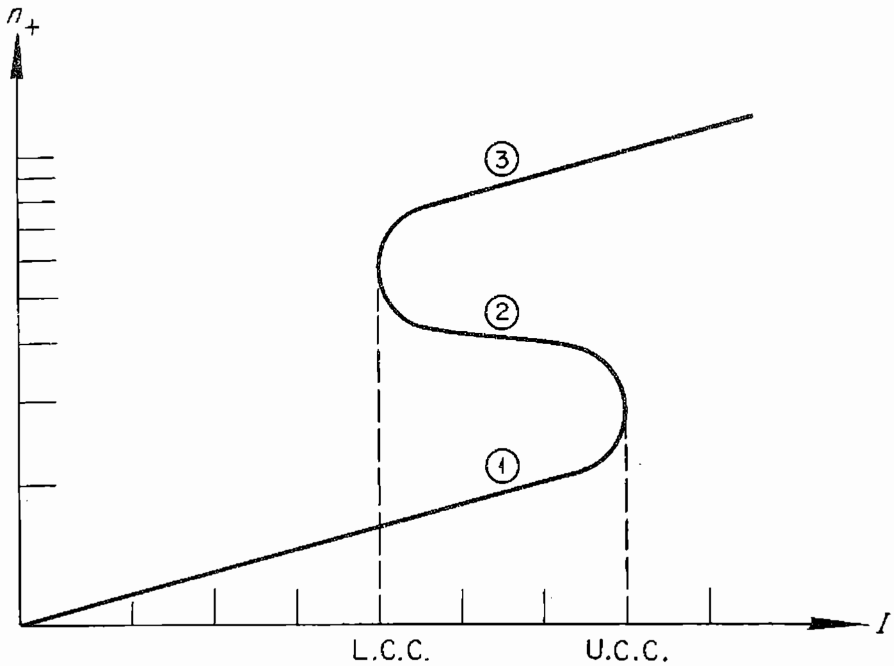

Fig. 1. Variation of Steady State Ion Density with Injected Current for an OGRA-Type Device.

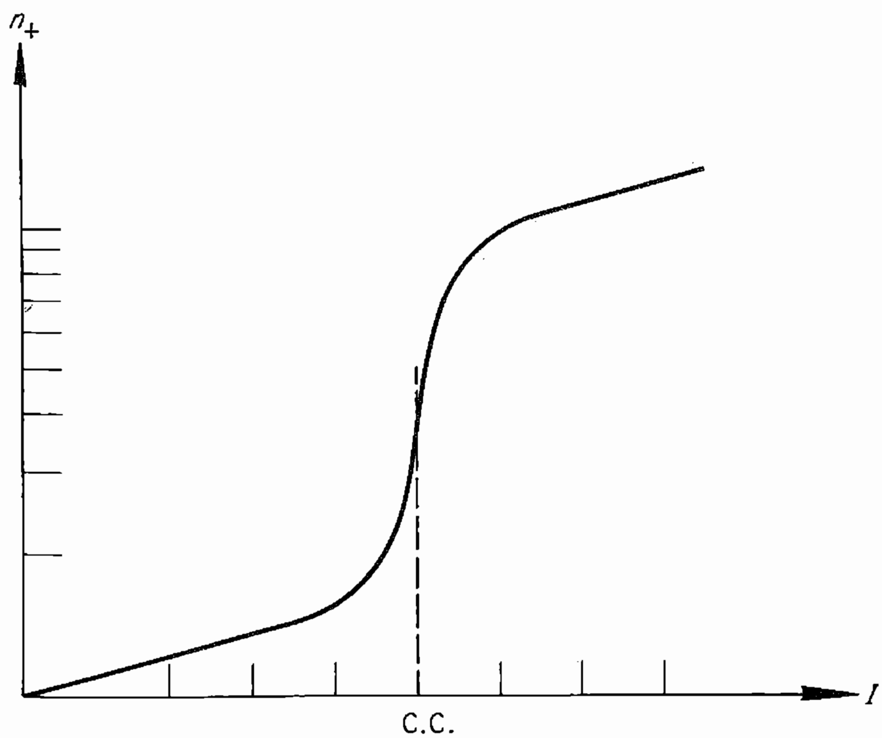

Fig. 2. Variation of Steady State Ion Density with Injected Current for a DCX-Type Device.

$$\Lambda = \left(\frac{\Gamma}{\theta} I + N_o\right) \left[1 + (\phi - 1) \frac{\bar{\ell} v \sigma_d^+}{v_o K}\right] \frac{1}{\phi} \tag{11}$$

The corresponding plasma density is $(\phi - 1)/K$. This equation for $I_{U.C.C.}$ can be solved quite readily by numerical means. It will be necessary to inject this U.C.C. in order to attain a high plasma density when the density is being increased from below.

In some cases there will not be a solution of the above equations, which means that burnout is not possible. The condition for which burnout is impossible is that

$$\frac{\Gamma \sigma_{cx}}{\sigma_d^+} \geq \sigma + \frac{\theta \bar{\ell}}{V v_o} \tag{12}$$

The approximations which are involved in the derivation of Eqs. (8) through (12) will break down if the resulting value of ϕ [as defined in Eq. (9)] is not larger than unity. In this case, no simple expression equivalent to Eq. (8) has been found and we must deal with the general solutions of Eq. (5). Numerical studies have so far indicated that the characteristic curve is still s-shaped in this region although it is much steeper and seems to be tending toward a DCX-shape.

A simple expression has also been derived for the lower critical current (L.C.C. in Fig. 1). This is

$$I_{L.C.C.} \cong \frac{\left(\frac{\Gamma}{\theta} I + N_o\right) \sigma_{cx} v V}{K \left[1 - \frac{3}{2\delta}\left(\frac{1}{\sigma_B^+ L} - \frac{1}{K}\right)\right]} \tag{13}$$

where

$$\delta = \left\{\left[\frac{I}{V} - \left(\frac{\Gamma}{\theta} I + N_o\right) \frac{\sigma_{cx} v}{K}\right] \frac{1}{3\sigma_c v P}\right\}^{1/2} \tag{14}$$

The corresponding plasma density is given by δ. Once a high plasma density has been obtained, it may be maintained by steady injection of a current larger than the L.C.C.

Details of these algebraic calculations and a summary of numerical results will be given in a future publication. I am greatly indebted to Drs. R. C. Gilbert and R. E. Hester for calling my attention to the possibility of multiple roots in the gas breakup case.

ABSOLUTE CONTAINMENT OF CHARGED PARTICLES IN A MAGNETIC FIELD

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Abstract.

In a magnetic field of the "mirror" type certain particles are "absolutely" contained irrespective of the constancy or otherwise of the magnetic moment. A criterion for absolute containment is derived and shown to ressemble that for containment on the adiabatic approximation.

Introduction.

The argument for containment of a particle in a magnetic mirror, in the absence of collisions, is normally based on the adiabatic invariance of the magnetic moment. This is not a true constant of the motion, and we expect arguments based on an adiabatic invariant to be valid only if the Larmor radius were very small compared to the dimensions of the field. There are, however, machines such as D.C.X. where this is certainly not true.

In this note, therefore, another principle of confinement is discussed called absolute containment, (1) which only depends on real constants of the motion. This principle is more restrictive than the adiabatic one but is complementary to it in that it applies when the Larmor radius is comparable to the field dimensions.

The condition for containment on the adiabatic approximation is

$$\frac{W}{W_{\underline{1}}} < \frac{B_m}{B_o}$$

where W is the total energy and $W_{\underline{1}}$ the component perpendicular to the field at the point where this has the value B_o.

(1) The existence of absolutely contained orbits has been noted by several workers notably at Livermore and Oak Ridge.

Absolute Containment

The motion of a charged particle in an axially symmetric field is governed by the Hamiltonian

$$H = \frac{1}{2m}\left[p_r^2 + p_z^2 + \left(\frac{p_\theta}{r} - \frac{eA}{c}\right)^2 \right]$$

where $p_r = m\dot{r}$, $p_z = m\dot{z}$, $p_\theta = mr^2\dot{\theta} + \frac{e}{c} r A$ and A is the θ component of the vector potential. It is convenient to introduce instead of A a stream function

$$\psi = \frac{e}{c} r A$$

then

$$\frac{e}{c} B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} \qquad \frac{e}{c} B_z = \frac{1}{r}\frac{d\psi}{\partial r}$$

so that ψ is a constant along a line of force.

To avoid confusion we shall consider a positive charged particle moving in a field such that B_z is also positive*. Then $\psi > 0$ and

$$H = \frac{1}{2m}\left[p_r^2 + p_z^2 + \left(\frac{p_\theta - \psi}{r}\right)^2 \right]$$

and

$$p_\theta = m\, r^2\, \dot{\theta} + \psi = L + \psi \text{ (say).}$$

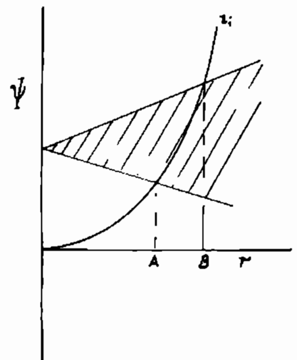
The particle is certainly restricted to that region in which the kinetic energy is positive i.e. to the region for which

$$| L_o + \psi_o - \psi | < \epsilon^{\frac{1}{2}} r \qquad (1)$$

where ϵ = 2mH and the subscript zero denotes initial values. We can therefore say, without invoking any approximations, that the particle is absolutely contained if Eq. 1 defines a closed region in the r, z plane.

There are several ways in which one can decide whether this criterion is satisfied but a useful one is the following graphical procedure. For any plane $z=z_i$ we draw $\psi(r,z_i)$ against r, and on the same diagram draw the region defined by Eq. 1. This region is bounded by straight lines and two distinct cases arise. If $(L_o + \psi_o) > 0$ then the region is as shown in Fig. 1, while if $(L_o + \psi_o) < 0$ it is as shown in Fig. 2. From this diagram one can immediately see what values of 'r' are permitted at any plane $z=z_i$ [e.g. in Fig. 1 the particle can range from r_A to r_B]. Thus we can construct the allowed regions in the r,z plane. It will be seen from Fig. 1 that if $L_o + \psi_o > 0$ then for every plane $z=z_i$ there will be some

* There is actually no loss of generality in this.



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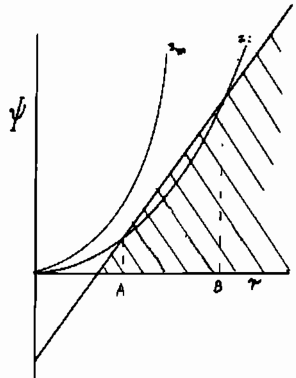

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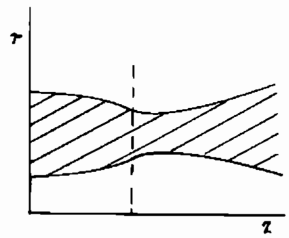

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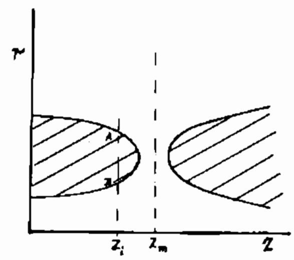

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allowed values of r, and the particle can always escape. Hence the first condition for absolute containment is

$$L_o + \psi_o < 0.$$

Physically this condition means that the particle orbit encircles the axis of symmetry.

On the other hand if $L_o + \psi_o < 0$ as in Fig. 2 then it may be that for some values of z_i (as e.g. z_m in figure) there are no allowed values of r. In this case the allowed region in the (r,z) plane is closed as in Fig. 4 and the particle is absolutely contained. The criterion for the particle to be unable to cross the plane $z=z_i$ is that the curves

$$\psi = \psi(r,z_i) \text{ and } \psi = L_o + \psi_o + \epsilon^{\frac{1}{2}} r$$

should not intersect.

This condition can be expressed, approximately, in a form similar to that obtained on the adiabatic approximation.

Suppose that the field has the usual 'mirror' shape, that the plane in which B_z reaches a maximum is $z=z_m$ and that on this plane B_z can be regarded as constant and equal to B_m. Then at the plane of the mirror

$$\psi(r,z) \approx \frac{e}{2c} B_m r^2$$

Similarly if z_o is the plane of injection

$$\psi_o \approx \frac{e}{2c} B_o r^2.$$

If we define the transverse kinetic energy with which the particle was injected* as

$$\frac{1}{2} m V_\theta^2 = W$$

then the criterion for absolute containment can be written in the form

$$\frac{W}{W_1} < \frac{4(\gamma - 1)}{\gamma^2} \frac{B_m}{B_o} \tag{2}$$

where γ is defined as $^{-L_o}/_{\psi_o}$.

Comparing this with the result obtained on the adiabatic approximation it will be seen that it is a more stringent condition since the expression

$$\frac{4(\gamma - 1)}{\gamma^2}$$

is always less than unity and attains this value only for $\gamma = 2$.

The condition $\gamma = 2$ corresponds to the case of a particle injected in such a way that the radius of injection is equal to its Larmor radius.

Conclusions

In a magnetic field certain particles are absolutely contained whether or not their magnetic moment can be regarded as constant. The orbits of these particles encircle the magnetic axis and satisfy a condition such as (2) which is generally more stringent than that deduced on the adiabatic approximation but reduces to it for particles which are injected so as to encircle the axis symmetrically.

The conditions we have found are <u>sufficient</u> to establish containment, it is interesting to speculate whether they are also <u>necessary</u>, in other words whether all particles which do <u>not</u> satisfy (1) actually escape. The Ergodic hypothesis would lead us to suppose that they would. However, unless we can estimate the time it takes for the particles to escape, this point is of only academic interest.

In most mirror machines the Larmor radius is small compared to any significant dimension and one would expect the adiabatic approximation to be a good one. However in D.C.X. the Larmor orbit is comparable with the dimensions and one might well doubt the validity of the adiabatic approximations. However it is just in this case that one can use the principle of absolute confinement which is in this sense complementary to the adiabatic approximation.

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* Note that the 'transverse' energy is now defined as that in the θ-direction rather than that perpendicular to the field.

ON PINCH STABILIZATION OVER LONG DURATION

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and

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Abstract

Long wave length perturbation modes of a linear pinch can be effectively stabilized through the use of a concentric conducting cylinder, In practice the finite conductivity of the cylinder prevents stabilization of slow perturbations. For the stabilization of these modes permanent diamagnets are required. Some methods are proposed for simulating such diamagnets with the help of liquid metallic walls in fast motion. Arrangements are shown for isolating static magnetic fields where the moving liquid metals perform the function of a diamagnet with $\mu = 0$.

Perturbations of long wave lengths in a pinched discharge are known to be stabilized by a conducting wall[1]. The working principle of this method can readily be seen in Fig. 1. When the discharge is displaced from its central position occupied in a., to that of b., the field in the vacuum becomes distorted while the magnetic field inside the condensor is "frozen". The distorted field exerts a force $\vec{F}$ on the discharge tending to push it back to the central equilibrium position.

As the conductivity σ is finite in any concrete case (except for superconductors, but they don't work in the presence of strong magnetic fields), it is obvious that this method works only for rapid perturbations, where the

1. R. J. Tayler, Proc. Phys. Soc. B. Vol. 70 1049(1957)

* This work was performed while in residence at the Israel Institute of Technology

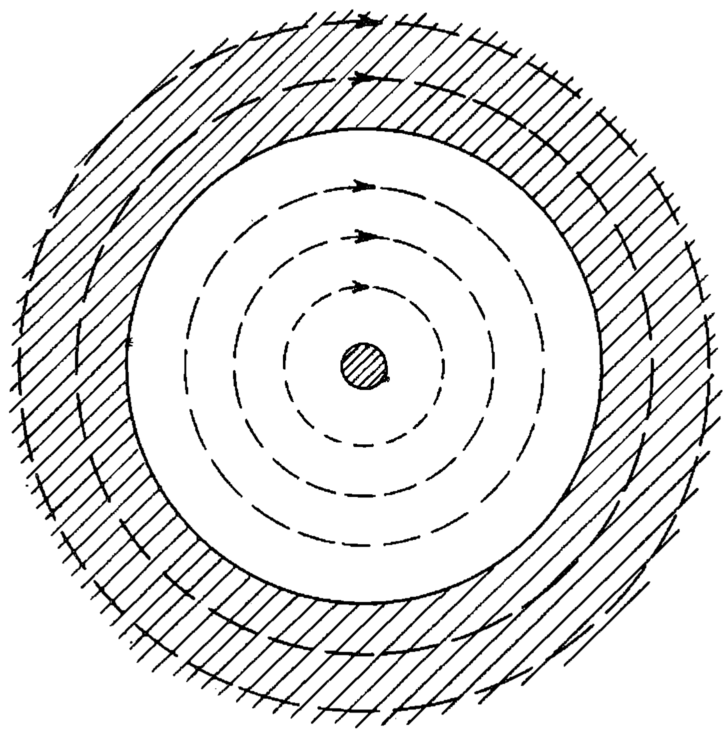

Figure 1a

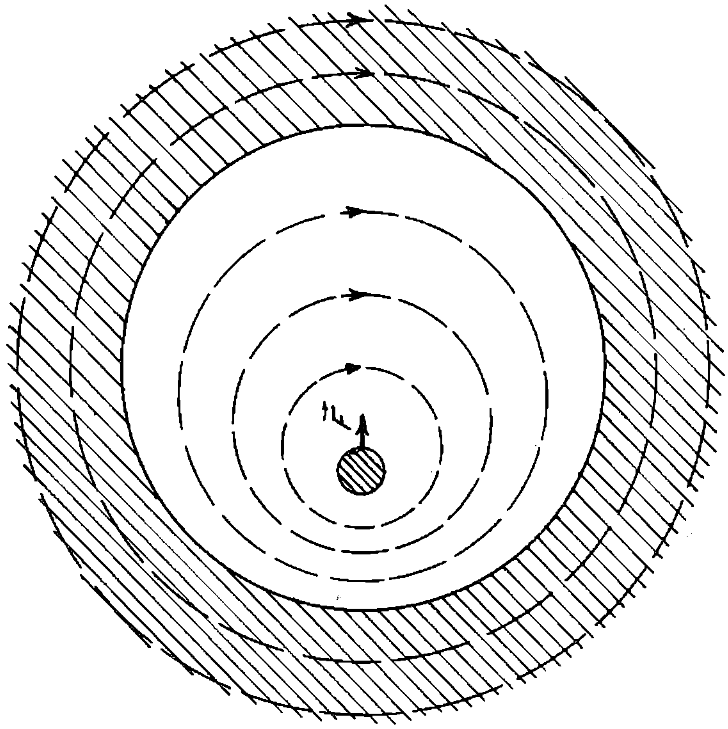

Figure 1b

characteristic time of perturbation τ is much smaller than the time needed for the perturbation of the field to penetrate the wall. This latter is of the order of $\tau_p \approx \mu_0 \sigma \delta^2$ where δ is a measure of the linear dimensions of the metal and μ_0 is the vacuum permeability[2] (MKS units are used). When $\tau >> \tau_p$ the perturbed field penetrates the wall unhindered and the stabilizing effect completely breaks down. As τ_p is only of the order of a few milliseconds in practical arrangements it is worth while to look for methods applicable to apparatus that might be designed for steady or quasisteady regime.

One method proposed here is based on the use of a diamagnetic wall. It is obvious (considering e.g. the mirror images) that a field is created tending to push the discharge to the central position, no matter how small the perturbation frequencies and velocities.

If we have for example, a wall made of an "ideal diamagnet" ($\mu = 0$. $B = 0$). The field of a discharge transplaced from the center will be as in Fig. 1b in the vacuum region, while the conductor remains field free. This field is indeed the unique solution of the $\nabla^2 \vec{B} = -\mu_0 \nabla \times \vec{i}$ equation in the vacuum region with the boundary conditions on the wall for the normal component $B_{n1} = B_{n2} = 0$ This solution is obviously independent of the history of former processes, in contrast with the case of an imperfectly conducting wall.

In nature - with the exception of superconductors - materials with very low susceptibilities only are known. Methods can be found however, to imitate diamagnetism in a certain sense. Two of them are outlined as follows:

1. One method is to set a conducting fluid in turbulent motion. As the field is frozen into the fluid the chaotic motion of its elements results in a destruction of the macroscopic time average field, as described by Landau and Lifschitz.[3] In a real case this effect is naturally limited by the finite conductivity of the fluid. The field is therefore not completely frozen and the diamagnetism is imperfect. Considering conditions for the setting in of turbulence it must be remembered, that besides the usual limitations (Reynolds number) the presence of a magnetic field imposes another one: the energy density of the magnetic field must be much smaller then the turbulent kinetic energy density.

2. The second method is to pump an originally field free conducting fluid sufficiently rapidly through the region concerned. If the time T in which a fluid element passes this region is much shorter than the penetration time τ_p the fluid remains field free. This means that in our example where $T = L/v$ (L is the characteristic flow length and v the fluid velocity): $v >> \frac{L}{\mu_0 \sigma \delta^2}$ Examples of possible arrangements are shown in Fig. 2. Calculating this relation for liquid sodium - taking $L = 1$ m $\delta = 0,1$ m - we obtain $v >> 8$ m/sec. Of course a sensible dia-

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2. Lyman Spitzer, Physics of Fully Ionized Gases - Interscience Publishers Inc. N.Y. 1956, p. 38

3. L. D. Landau and E. M. Lifschitz, Elektrodinamica Sploshnich Sred, page 302, Gosudarstvenoe Izdatelstvo Tehniko Teoretitcheskoi Literatury, Moscow 1957

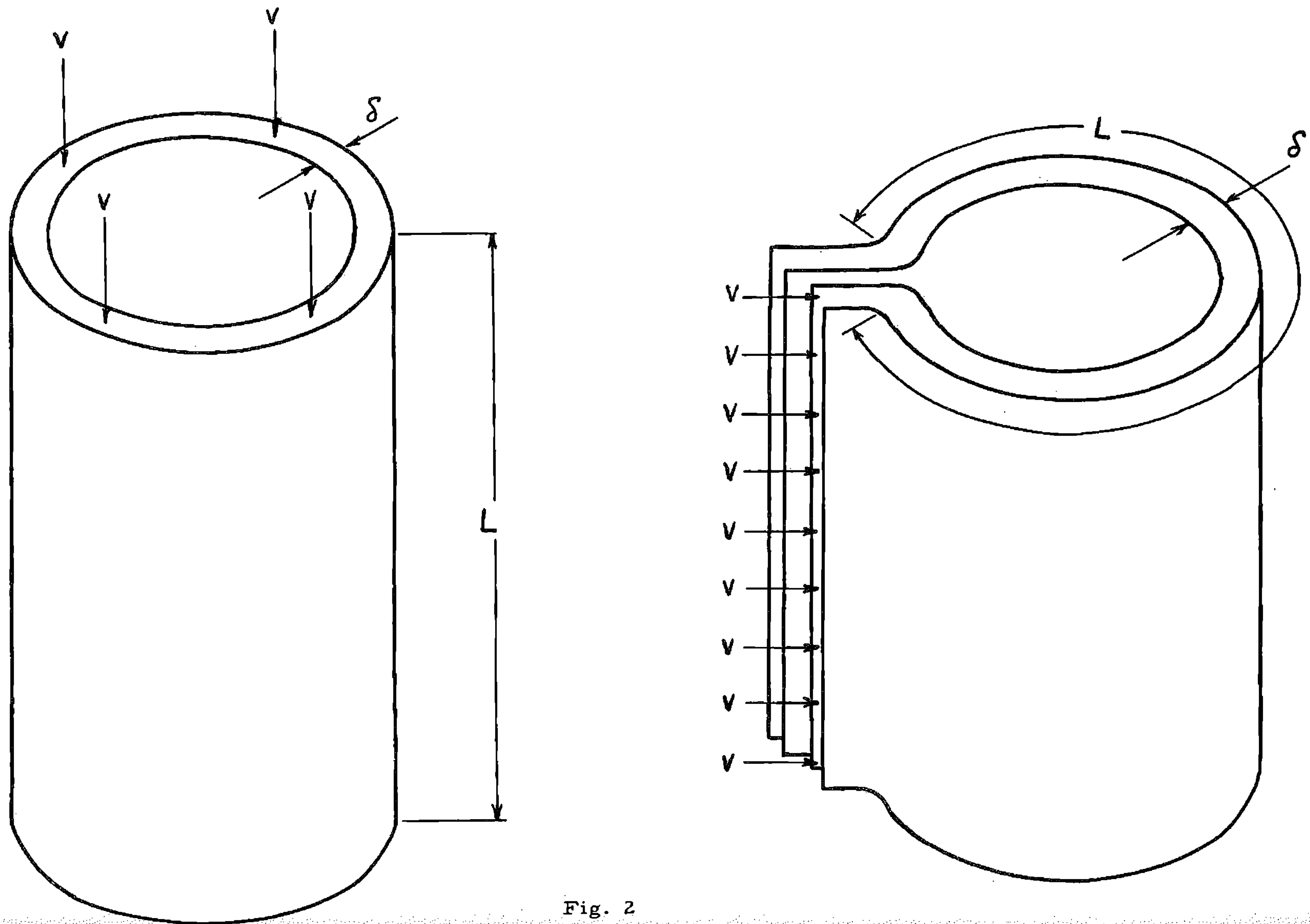

Fig. 2

magnetic effect can be obtained by $\nu \simeq \frac{1}{\mu_0 \sigma \delta^2}$. In a practical case the stabilizing fluid might act as a coolant too.

In principle other methods to push the pinch back to the center can be found. This is the case for example for a rotating conducting wall. It can be shown that in the case where the field of the perturbed pinch penetrates the wall this exhibits a tendency to drag along the pinch in its rotation. If a phase lag exists a force acting on the pinch is created having a component towards the center.

BOUNDARY LAYER FORMATION IN THE PINCH

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I. INTRODUCTION

Containment of plasmas may be achieved by either vacuum magnetic fields or pinch magnetic fields. Vacuum magnetic fields are created by external coils, whereas in the pinch device the primary current is induced in the plasma, and without this current there exists no containing field. The current and magnetic field spatial distributions are of interest since the degree of stability of the pinch plasma column is dependent on the sharpness of the boundary.

In this paper processes that determine the pinch current's spatial distribution for times prior to the implosion of a deuterium plasma are examined. A one dimensional problem is treated where there is an externally applied stabilizing magnetic field in the direction of the electric field which immobilizes the charged particles in the plasma. It is assumed that this field is of such a magnitude as to make the heat and charged particle diffusion terms across the field negligible. In this manner, wall effects are also conveniently eliminated. Further, the strength of the self or pinching magnetic field (component of the magnetic field normal to the electric field) is considered to be be small relative to the stabilizing magnetic field over the interval of time for which the results are significant. Hence, mass motion of the plasma is ignored. As a result of these assumptions, the stabilizing field does not appear explicitly in the calculation. The validity of these assumptions is examined in the disucussion of the results.

The plasma equations of Wyld and Watson[1] are generalized to include spatial dependence. The different particles are treated as having local Maxwellian

[1] Wyld, N. W. and Watson, K. M., Gatlinburg Controlled Fusion Conference Proceedings (June 4-7, 1956)T.I.D. 7520 .

distributions, i.e., temperatures are assigned. This is a limitation on this work since for the particle densities and electric field strengths utilized in pinch work some electrons should tend to "run-away". However, should this effect result only in a reduction of the effective resistance then within the framework of this calculation the skin depths should be smaller than those calculated. In addition to the electron and ion temperatures the variables of the problem are the percentage of ionization, the resistivity of the gas, and the current density. Since the current and resistivity are interdependent the plasma equations are coupled to the field equations.

At the boundary of the plasma the electric field is set equal to a constant applied electric field minus a self induced electric field. It is also assumed that the current density and its derivative go to zero at large distances from the plasma boundary. Some physical situations for which this model is applicable are discussed along with the results.

The computations were performed on the I.B.M. 704 at Livermore.

II. FIELD EQUATIONS

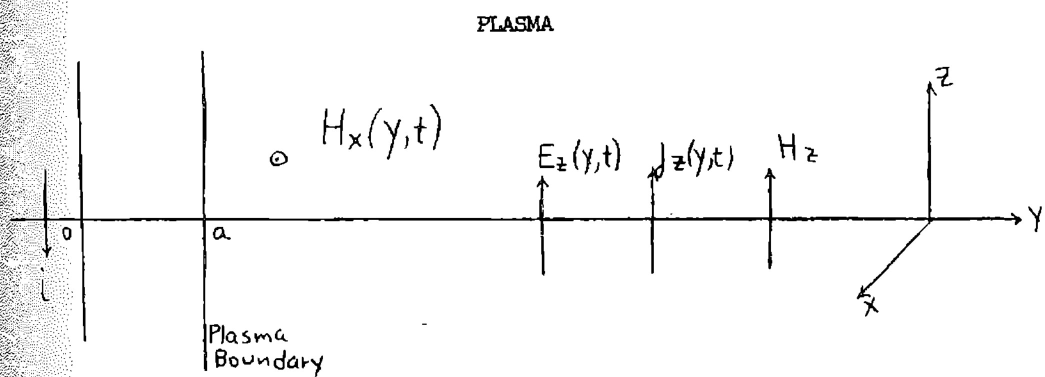


Fig. 1 The Field Variables

The problem considered (see fig. 1) is one dimensional, i.e., the electric field, $\vec{E}$, the magnetic field, $\vec{H}$, and the current density, $\vec{j}$, are functions of y and t. The x component of the magnetic field, $H_x(y,t)$, is the self magnetic field which results from the current density $j_z(y,t)$; the z component of the magnetic field, H_z, is the externally applied constant stabilizing field. The field equations are then

$$\frac{\partial E_z}{\partial y} = -\frac{1}{c}\frac{\partial B_x}{\partial t}, \quad a \leq y < \infty , \tag{1}$$

$$\frac{\partial H_x}{\partial y} = -\frac{4\pi}{c} j_z , \quad a \leq y < \infty , \tag{2}$$

$$E_z = \eta_z \, j_z \, , \quad a \le y < \infty \, , \tag{3}$$

where $\eta\,(y,t)$ is the resistivity of the plasma.

Differentiating Eq. (3) with respect to y gives (let $j = j_z$ and $\eta = \eta_z$)

$$\frac{\partial E_z}{\partial y} = \frac{\partial}{\partial y}\,(\eta j).$$

Then from Eq. (1):

$$-\frac{1}{c}\,\frac{\partial B_x}{\partial t} = \frac{\partial}{\partial y}\,(\eta j)\;.$$

Now differentiating the above yields

$$-\frac{1}{c}\,\frac{\partial^2 B_x}{\partial y\,\partial t} = \frac{\partial^2}{\partial y^2}\,(\eta j)$$

Differentiating Eq. (2) with respect to t gives

$$\frac{\partial^2 H_x}{\partial t\,\partial y} = -\frac{4\pi}{c}\,\frac{\partial j}{\partial t}\;.$$

Using $B_x = \mu\, H_x$ then gives

$$\frac{\partial j}{\partial t} = \frac{c^2}{4\pi\mu}\,\frac{\partial^2}{\partial y^2}\,(\eta j) \tag{4}$$

for $a \le y < \infty$, $t \ge 0$.

Equation (4) is the differential equation to be solved for the current density, $j\,(y,t)$. It is a nonlinear equation since η depends on j through the plasma equations.

The boundary conditions which are used for Eq. (4) are

$$j \to 0 \text{ as } y \to \infty\,, \tag{5}$$

and $$E_o - L\,\frac{di}{dt} = \eta\,(a,t)\; j\,(a,t)\,, \tag{6}$$

where E_o is a constant applied electric field,

L is a constant inductance,

and i is the total current, i.e.,

$$i = \int_a^\infty j\, d\, y. \tag{7}$$

Eq. 6 may be rewritten as

$$E_o - L\,\frac{d}{dt}\int_a^\infty j\, d\, y = \eta\,(a,t)\; j\,(a,t)\,,$$

or

$$E_o - L\int_a^\infty \frac{\partial j}{\partial t}\, d\, y = \eta\,(a,t)\; j\,(a,t).$$

Substituting Eq. (4) into the above equation

$$E_o - \frac{\mathcal{L}\, c^2}{4\pi\mu} \int_a^\infty \frac{\partial^2}{\partial y^2} (\eta j)\, d\, y = \eta\,(a,t)\; j\,(a,t),$$

or

$$E_o + \frac{\mathcal{L}\, c^2}{4\pi\mu} \left.\frac{\partial(\eta j)}{\partial y}\right|_{y=a} = \eta\,(a,t)\; j\,(a,t),$$

where

$$\frac{\partial(\eta\, j)}{\partial\, y} \rightarrow 0 \quad \text{as } y \rightarrow \infty$$

Let

$$a = \frac{\mathcal{L} c^2}{4\pi\mu} \tag{8}$$

then the boundary condition becomes

$$E_o = \eta\,(a,t)\; j\,(a,t) - a \left.\frac{\partial(\eta j)}{\partial y}\right|_{y\,=\,a} . \tag{9}$$

As an initial condition the resistivity of the gas is taken to be uniform, i.e., $\eta\,(y,0) = \eta\,(0)$. The initial current density may then be taken as zero, or an initial current density may be chosen satisfies the conditions as $y \rightarrow \infty$; e.g.,

$$j\,(y,0) = j_o\, e^{-(y-a)/\delta} \tag{10}$$

and to be consistent with the boundary condition Eq. (8) we have

$$j_o = \frac{E_o}{\eta(0)}\; \frac{1}{1 + \frac{a}{\delta}} . \tag{11}$$

III. PLASMA EQUATIONS

The energy balance equation is

$$\eta\, j^2 = \alpha\; \epsilon_o\, \frac{\partial n_e}{\partial t} + \frac{\partial}{\partial t}\left[\frac{3}{2}\, n_e\, \theta_e + \frac{3}{2}\, n_i\, \theta_i\right], \tag{12}$$

where $\eta\,(y,t)$ is the resistivity of the plasma,

$j\,(y,t)$ is the current density,

ϵ_o is an average ionization potential (16.2 ev),

$\alpha\, \epsilon_o$ is the average energy expended per ion pair (ion plus electron)

$n_e\,(y,t) = n_i(y,t)$, the electron and ion densities,

$\theta_e\,(y,t) = k\, T_e$, the electron temperature, and

$\theta_i\,(y,t) = k\, T_i$, the ion temperature.

The left hand side of Eq. 12 is the rate of ohmic heating per unit volume. It is assumed that this energy goes into the ionization of neutrals, and heating of the charged particles. The first term on the right hand side of Eq. 12 is the energy expended on ionization. It essentially represents an energy drain on the electrons. Now the ionization of the deuterium molecule may proceed in various ways, and in general is not a one step process. However, this complexity is avoided by treating the deuterium gas as monatomic with an average ionization potential, ϵ_o. The cross section for ionization is taken to be the same as for the ionization of D_2. Energy losses, such as inelastic collisions of electrons with neutrals, are included in the factor α. For example, if on the average α' excitation collisions occur for each ionization collision with the same dissipation of energy, then the rate of this energy loss per unit volume is $\alpha' \epsilon_o \frac{\partial n_e}{\partial t}$. We choose $\alpha = 2$. (two) .

Because of the mobility of the electrons, their temperature increases much faster under the influence of the electric field than that of the ions. For electric fields of the magnitude employed in pinches the electrons will rapidly reach a kinetic temperature corresponding to the ionization potential. Because of the mass difference, the electrons give up little energy to the neutrals during their lifetime, i.e., the ionization time is short as compared to an energy exchange time. If is for this reason that the change in temperature of the neutrals is not included in Eq. 12. At the completion of ionization the electrons lose energy primarily by transferring it to the ions. It is assumed that the Bremstrahlung Radiation is negligible for the electron temperatures achieved during the time before the implosion. Charge exchange effects should also be small for the times of interest at the densities and energies for this problem.

The resistivity of the partially ionized plasma is

$$\eta = \frac{m}{n_e e^2} (\nu_i + \nu_n), \tag{13}$$

where m is the electron mass,

e is the electron charge

ν_i is the collision frequency of electrons with ions, and

ν_n is the collision frequency of electrons with neutrals.

Electron collisions with neutrals are retained in the definition of electrical resistivity since the collisions impede the motion of the electrons in the direction of the electric field.

The effective ν_i is

$$\nu_i = \left[\frac{\pi^{3/2}}{4(0.582)} \frac{e^4}{\sqrt{2m}} \ln\left(\frac{b_{max}}{b_{min}}\right)\right] \frac{n_i}{\theta_e^{3/2}} , \qquad (14)$$

where the factor 0.582 accounts for electron-electron collisions,[2] and

$$\nu_n = (n_o - n_e) \left(\frac{3\,\theta_e}{m}\right)^{1/2} \sigma , \qquad (15)$$

where n_o = gas density,

and σ = cross section for electron collisions with neutrals.

We take $\sigma = 3 \times 10^{-16}$ cm^2.

Note that the resistivity as defined does not depend on the ion density, n_i, or temperature, θ_i, but only on the electron temperature and the fraction of ionization. The neutral particles are treated as stationary targets (i.e., zero temperature).

The ionization equation is

$$\frac{\partial n_e}{\partial t} = n_e (n_o - n_e) \overline{\sigma_i v_e} , \qquad (16)$$

where σ_i is the ionization cross section, and v_e is the electron velocity.

The ionization cross section is obtained by fitting published experimental[3] results.

$$\sigma_i = 0, \quad \epsilon < \epsilon_o = (16.2)(1.6 \times 10^{-12}) \text{ ergs}$$

$$\sigma_i = 1.20 - \frac{4}{30}\left(\frac{\epsilon}{\epsilon_o} - 4\right)^2 , \quad \epsilon_o < \epsilon < 4\,\epsilon_o$$

$$\sigma_i = 1.36 - 0.04\frac{\epsilon}{\epsilon_o} , \quad 4\,\epsilon_o < \epsilon < 20\,\epsilon_o$$

$$\sigma_i = 0.56 , \quad \epsilon > 20\,\epsilon_o . \qquad (17)$$

where σ_i is expressed in units of πa_o^2.

The quantity $\overline{\sigma_i v_e}$ is obtained by averaging over a Maxwell-Boltzmann distribution.

Then

$$\overline{\sigma_i v} = -2\, a_o^2 \left(2\pi \frac{\theta_e}{m}\right)^{\frac{1}{2}} \Big[\left(-0.8 - \frac{4}{3}\frac{1}{\alpha} + 0.8\frac{1}{\alpha^2}\right)e^{-\alpha}$$
$$+ \left(0.16 - 0.987\frac{1}{\alpha} - 0.8\frac{1}{\alpha^2}\right)e^{-4\alpha} + \left(-0.8 - 0.08\frac{1}{\alpha}\right)e^{-20\alpha}\Big], \qquad (18)$$

where $\alpha = \dfrac{\epsilon_o}{\theta_e}$.

---

[2] See Spitzer, L. Physics of Fully Ionized Gases, p. 84

[3] e.g. Mott, N. F. and Massey, H.S.W., The Theory of Atomic Collisions, Oxford Press, Page 245, 1959

The equation giving the rate at which energy is transferred from electrons to ions is

$$\frac{\partial \theta_i}{\partial t} = \frac{8}{3}\sqrt{2\pi}\, e^4\, n_e \frac{\sqrt{m}}{M} \frac{\theta_e - \theta_i}{\theta_e^{3/2}} \ln\left(\frac{b_{max}}{b_{min}}\right), \tag{19}$$

where M is the ion mass.

IV SUMMARY AND METHOD OF SOLUTION

Let $f = \frac{n_e}{n_o} = \frac{n_i}{n_o}$, and let θ_e, θ_i be expressed in units of ϵ_o, the ionization potential. Let η be in units of $\eta_o = \frac{4\pi\mu}{c^2}$, then we can summarize the equations to be solved

$$\frac{\partial j}{\partial t} = \frac{\partial^2(\eta j)}{\partial y^2} \tag{20}$$

$$A_6 E_o = \eta(a,t)\, j(a,t) - a \left.\frac{\partial(\eta j)}{\partial y}\right|_{y=a} \tag{21}$$

$$\eta = A_2 \left(\frac{1}{f} - 1\right) \theta_e^{1/2} + A_3 \theta_e^{-\frac{3}{2}} \tag{22}$$

$$\frac{\partial \theta_e}{\partial t} = -\frac{\partial \theta_i}{\partial t} - \left(\frac{2\alpha}{3} + \theta_e + \theta_i\right) \frac{1}{f}\frac{\partial f}{\partial t} + A_1 \frac{1}{f}\, \eta\, j^2 \tag{23}$$

$$\frac{\partial \theta_i}{\partial t} = A_5 f \theta_e^{-3/2} (\theta_e - \theta_i) \tag{24}$$

$$\begin{aligned}\frac{\partial f}{\partial t} = A_4 f(1-f)\, \theta_e^{1/2} \Big[&\left(0.8 + \tfrac{4}{3}\theta_e - 0.8\,\theta_e^2\right) e^{-\frac{1}{\theta_e}} \\ &+ \left(-0.16 + 0.987\,\theta_e + 0.8\,\theta_e^2\right) e^{-\frac{4}{\theta_e}} \\ &+ \left(0.8 + 0.08\,\theta_e\right) e^{-\frac{20}{\theta_e}} \Big]\end{aligned} \tag{25}$$

where the constants are

$$A_1 = \frac{2}{3} \frac{\eta_o}{n_o \epsilon_o}$$

$$A_2 = \frac{\sigma}{\eta_o e^2} \sqrt{3 m \epsilon_o}$$

$$A_3 = \left(\frac{\pi}{2\epsilon_o}\right)^{3/2} \sqrt{m} \frac{e^2}{2(0.582)\eta_o} \ln \frac{b_{max}}{b_{min}}$$

$$A_4 = 2\, a_o^2\, n_o \sqrt{\frac{2\pi\epsilon_o}{m}}$$

$$A_5 = \frac{8}{3}\sqrt{\frac{2\pi m}{M}}\; n_o\, e^4\, \epsilon_o^{-\frac{3}{2}} \ln \frac{b_{max}}{b_{min}}$$

$$A_6 = \frac{1}{\eta}\,.$$

The solutions of Eq. (22) - (25) with j = constant is the problem considered in the paper of Wyld and Watson[1]. The resulting set of ordinary differential equations can be solved numerically,

If we let η = constant, Eq. (20) becomes

$$\frac{\partial j}{\partial t} = \eta \frac{\partial^2 j}{\partial y^2}\quad,\quad a \le y < \infty\quad,\quad t \ge 0;$$

with boundary conditions

$$j\,(a,t) - a \left.\frac{\partial j}{\partial y}\right|_{y=a} = \frac{E_o}{\eta}$$

$$j\,(y,t) \to 0 \quad \text{as } y \to \infty$$

and initial condition

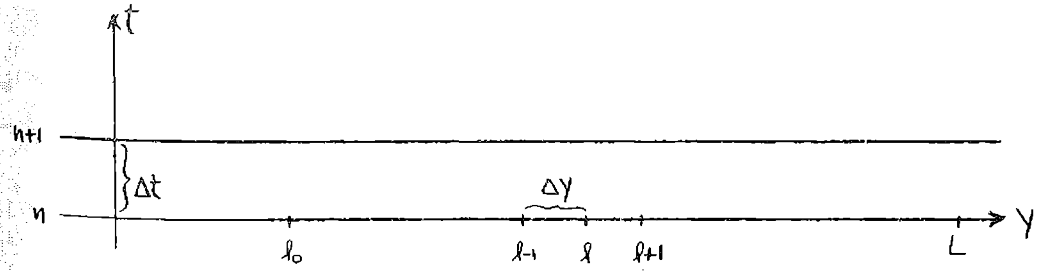
$$j\,(y,\,0) = j_o\, e^{-\frac{y-a}{\delta}}$$

where
$$j_o\left(1 + \frac{a}{\delta}\right) = \frac{E_o}{\eta}$$

This problem can be solved by using the Laplace transformation and the solution is

$$j\,(y,\,t) = \frac{E_o}{\eta}\left[1 - \frac{\eta}{\delta^2}\,\frac{\sqrt{\eta}}{a}\,\frac{2}{\pi}\int_0^\infty \frac{e^{-tx^2}\left[\frac{\sqrt{\eta}}{a}\sin\frac{y-a}{\sqrt{\eta}}\,x + x\cos\frac{y-a}{\sqrt{\eta}}\,x\right]dx}{x\,(x^2 + \frac{\eta}{\delta^2})\,(x^2 + \frac{\eta}{a^2})}\right]$$

The above solution to this special case can be evaluated in terms of error functions and is quite useful in checking the results in the early stages of the more general calculation. In order to solve the complete system of equations (20) - (25), we solve a set of finite difference equations. Consider the mesh



where $a = l_o \Delta y$

The implicit difference equation corresponding to (20) is

$$j_\ell^{n+1} - j_\ell^{n} = \frac{\Delta t}{2(\Delta y)^2}\left[\eta_{\ell+1}^{n+1}\, j_{\ell+1}^{n+1} - 2\eta_\ell^{n+1}\, j_\ell^{n+1} + \eta_{\ell-1}^{n+1}\, j_{\ell-1}^{n+1}\right]$$

$$+ \frac{\Delta t}{2(\Delta y)^2}\left[\eta_{\ell+1}^{n}\, j_{\ell+1}^{n} - 2\,\eta_\ell^{n}\, j_\ell^{n} + \eta_{\ell-1}^{n}\, j_{\ell-1}^{n}\right],$$

$$\ell = \ell_o + 1\,,\ \ldots,\ L - 1\ .$$

The boundary condition at $y = a$ becomes

$$E_o = \eta_{\ell_o}^{n+1}\, j_{\ell_o}^{n+1} - \ell_o\left(\eta_{\ell_o+1}^{n+1}\, j_{\ell_o+1}^{n+1} - \eta_{\ell_o}^{n+1}\, j_{\ell_o}^{n+1}\right)$$

The maximum value of y in the mesh is $L\Delta y$. In order to treat the imposed conditions on j as $y \to \infty$, j_L is set equal to zero, but the distance $L\Delta y$ must be taken large enough so that the solution is not affected by changes in the cut-off. The cut-off distance is a result of numerical experimentation and differs with the various cases presented. The method of solution of the implicit difference equations is given by Richtmyer.[4]

The Eqs. (23) - (25) must be solved at each value of ℓ in the mesh. Since they do not contain spatial derivatives, they can be considered as ordinary differential equations and solved by a standard method for each ℓ.

V. RESULTS AND MODELS

It is known that in addition to an externally applied B_z, a surrounding conducting wall enhances the stability of a plasma column. Suppose in Fig. 1 that the plane at $y = 0$ is a conductor. If the region $0 \leq y \leq a$ is occupied by a dielectric, and a plasma occupies the region $y \geq a$ then a charge density is deposited on the surface of the dielectric which neutralizes the polarization charge density and the electric field lines in the plasma then do not terminate on the surface, but run in the z direction as assumed in the model.

We can derive the boundary condition at $y = a$ under these conditions. Integrating Eq. (1) with respect to y gives

$$E_z\,(y,\,t) - E_o = -\,\frac{a}{c}\,\frac{\partial}{\partial t}\,B_x\,(a,\,t) - \frac{1}{c}\,\frac{\partial}{\partial t}\int_a^y B_x\,(\zeta,\,t)\,d\,\zeta$$

[4] R. D. Richtmyer, Difference Methods for Initial-Value Problems, Interscience, N. Y. page 101.

We must have $E_z(y, t) \to 0$ as $y \to \infty$, so

$$E_o = \frac{a}{c} \frac{\partial}{\partial t} B_x(a, t) - \frac{1}{c} \frac{\partial}{\partial t} \int_a^{\infty} B_x(y, t)\, dy \qquad (26)$$

Integrating Eq. (2) gives

$$H_x(y, t) - H_x(a, t) = -\frac{4\pi}{c} \int_a^{y} j(\zeta, t)\, d\zeta$$

We must have $H_x(y,t) \to 0$ as $y \to \infty$, so

$$H_x(a, t) = \frac{4\pi}{c} \int_a^{\infty} j(y, t)\, dy \qquad (27)$$

Hence

$$H_x(y, t) = \frac{4\pi}{c} \int_a^{\infty} j(y, t)\, dy - \frac{4\pi}{c} \int_a^{y} j(\zeta, t)\, d\zeta,$$

or

$$H_x(y, t) = \frac{4\pi}{c} \int_y^{\infty} j(\zeta, t)\, d\zeta \qquad (28)$$

Using Eqs. (27) and (28) in (26) we have

$$E_o = \frac{4\pi a\mu}{c^2} \frac{\partial}{\partial t} \int_a^{\infty} j(\zeta, t)\, d\zeta + \frac{4\pi\mu}{c^2} \frac{\partial}{\partial t} \int_a^{\infty} dy \int_y^{\infty} j(\zeta, t)\, d\zeta$$

Integrating the second term by parts we have

$$E_o = \frac{4\pi\mu}{c^2} \frac{\partial}{\partial t} \left[a \int_a^{\infty} j(\zeta, t)\, d\zeta - a \int_a^{\infty} j(\zeta, t)\, d\zeta + \int_a^{\infty} y\, j(y, t)\, dy \right]$$

or

$$E_o = \frac{4\pi\mu}{c^2} \int_a^{\infty} y \frac{\partial j}{\partial t}\, dy \qquad (29)$$

Now using Eq. (4) in Eq. (29) we have

$$E_o = \int_a^{\infty} y \frac{\partial^2 (\eta j)}{\partial y^2}\, dy$$

Integrating by parts we have

$$E_o = \left[y \frac{\partial(\eta j)}{\partial y} \right]_a^{\infty} - \int_a^{\infty} \frac{\partial(\eta j)}{\partial y}\, dy$$

We have assumed $\eta(y, t)\, j(y, t) \to 0$ as $y \to \infty$,

and if we assume $y \dfrac{\partial(\eta j)}{\partial y} \to 0$ as $y \to \infty$;

then

$$E_o = \eta (a, t) j (a, t) - a \left. \frac{\partial (\eta j)}{\partial y} \right|_{y = a}$$

is the boundary condition at $y = a$. This corresponds to Eq. (9), where a is now taken to be the thickness of an insulating region between the conducting shell and the plasma. This distance fixes the inductance in Eq. (6) by using Eq. (8).

For an $a = 1$ cm. we present two cases of interest. The first case is for an applied electric field $E_o = 100$ volts/cm, and the gas density $n_o = 10^{15}/cm^3$. This case corresponds to pinch devices at Livermore and Los Alamos. In Fig. (3) the values of the plasma and field variables at the plasma boundary are plotted as a function of time. In Figs. (4) and (5), $j (y, t)$ and $H_x (y, t)$ are plotted as a function of distance from the wall for successive times. In Fig. (6) the plasma variables are plotted as a function of distance from the wall at a time corresponding to completion of ionization at the plasma boundary.

The second case for $a = 1$ cm. is for an $E_o = 2$ volt/cm and $n_o = 10^{13}/cm^3$. This case is intended to correspond to a large pinch device such as Zeta at Harwell. Similar results for this case are presented in Figs. (11) - (14).

Another physical situation for which the model applies is the following

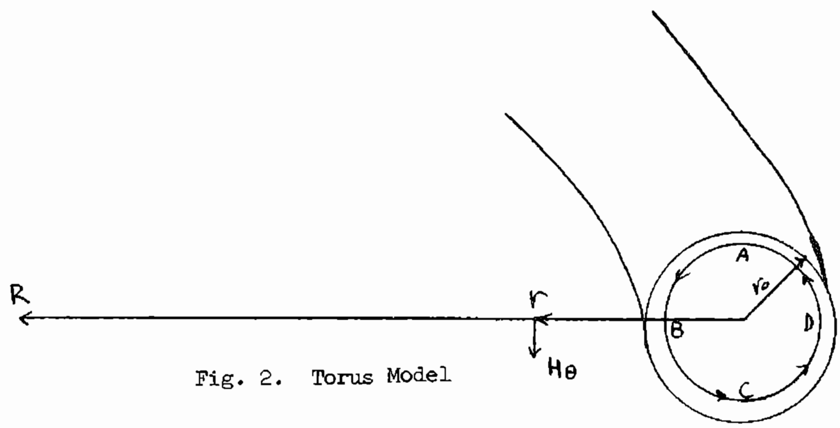


Fig. 2. Torus Model

Consider a torus with a non-conducting shell; and assume that the minor radius, r_o, is small compared to the major radius, R, and that the current skin depth, δ, is small as compared to the minor radius; i.e. $\frac{r_o}{R} << 1$; $\frac{\delta}{r_o} << 1$.

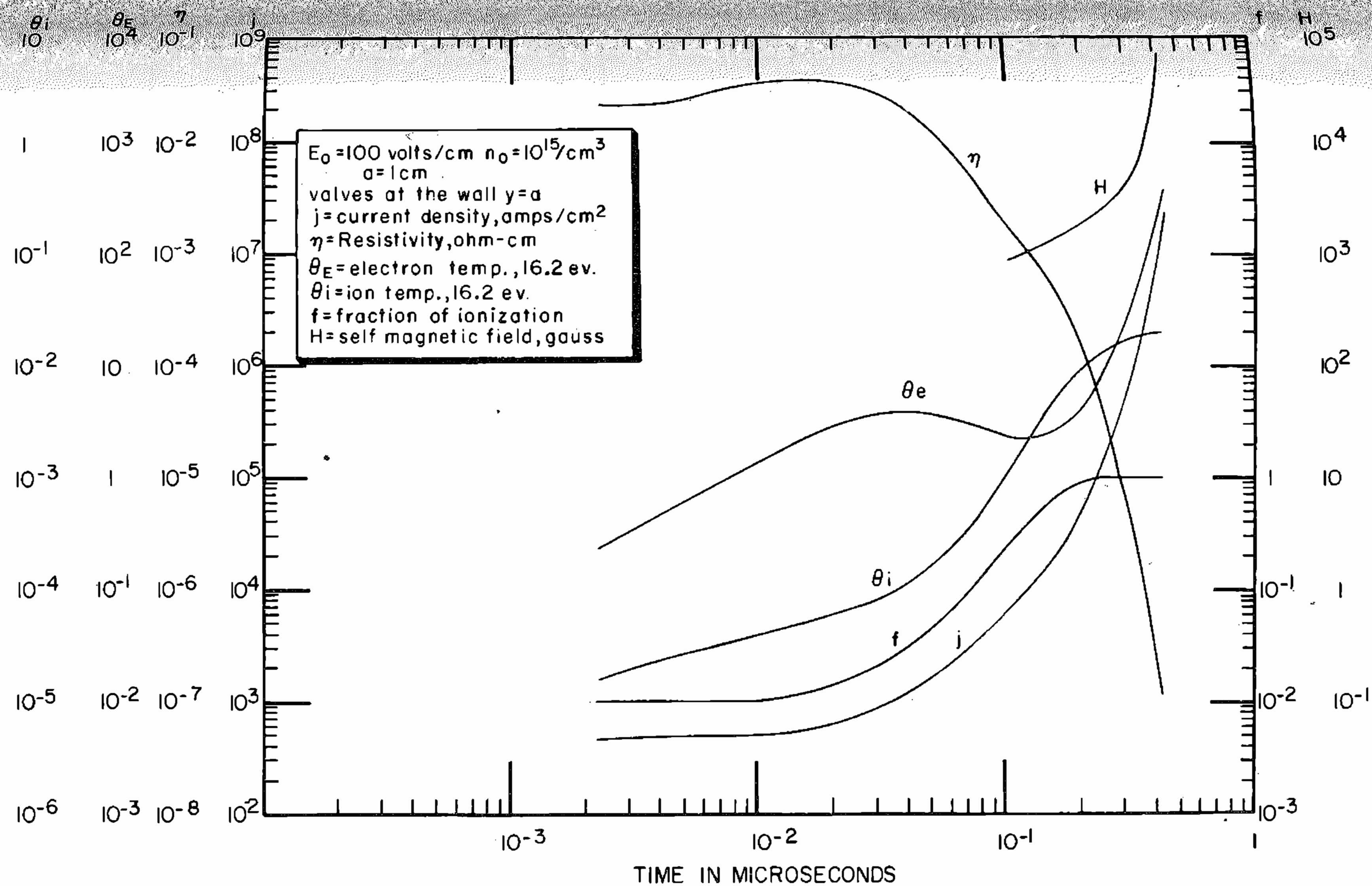
Integrating $\vec{\nabla} \times \vec{H} = \frac{4\pi}{c} \vec{j}$ over the area enclosed by the curve A B C D, which is within the current layer, gives

$$\oint \vec{H} \cdot d\vec{s} = \frac{4\pi}{c} \int_0^{r_o} 2 \pi r j d r$$

or

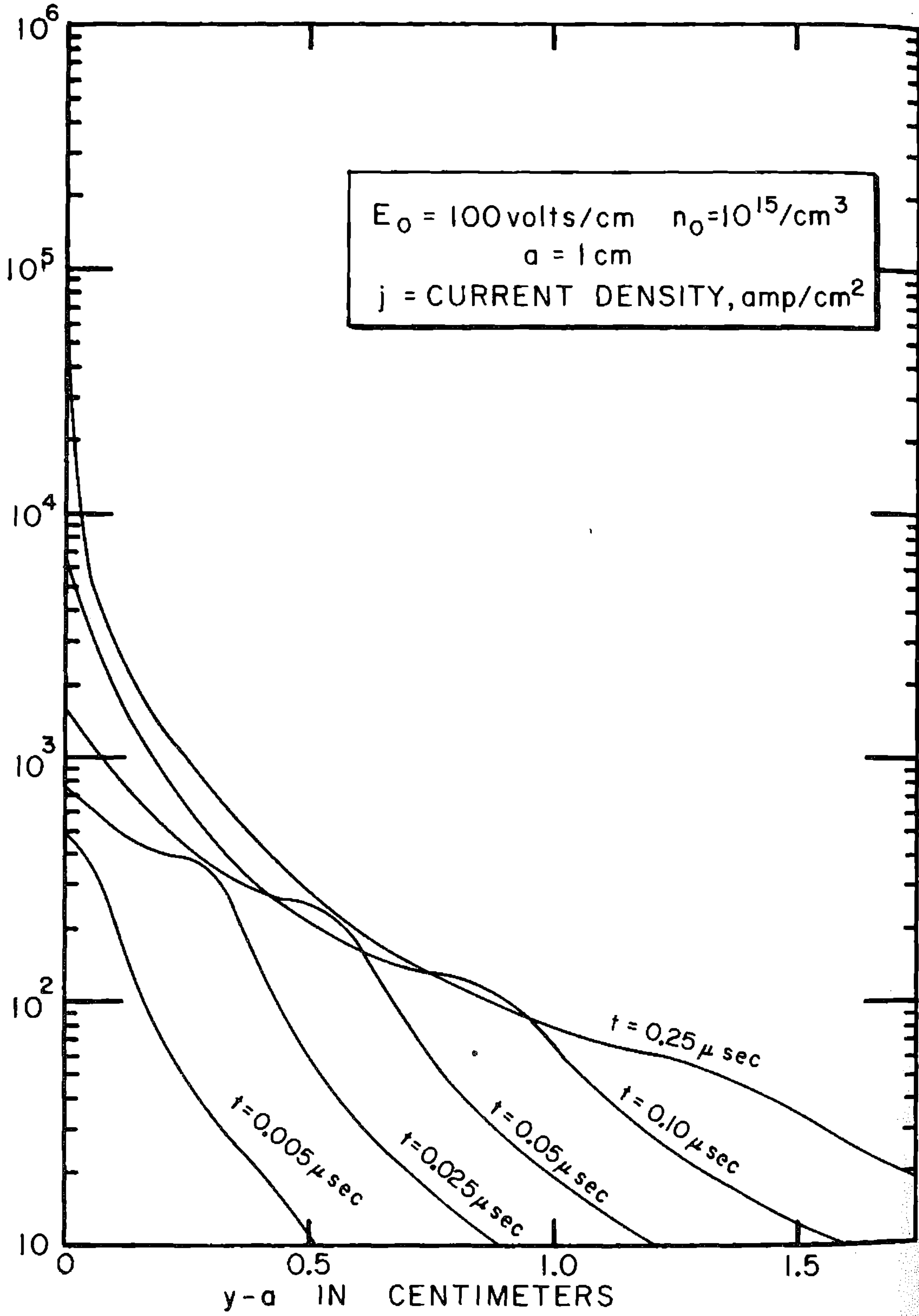
$$2 \pi r_o H_\theta = \frac{4\pi}{c} 2 \pi r_o \int_0^{r_o} j d r,$$ using the above assumption.

(Text continues on page 97)



E_0 = 100 volts/cm $n_0 = 10^{15}/cm^3$
a = 1 cm
valves at the wall y = a
j = current density, amps/cm^2
η = Resistivity, ohm-cm
θ_E = electron temp., 16.2 ev.
θ_i = ion temp., 16.2 ev.
f = fraction of ionization
H = self magnetic field, gauss
θ_i
θ_E
η
j
f
H
θ_e
TIME IN MICROSECONDS
10^{-3}
10^{-2}
10^{-1}
1

Figure 3



E$_0$ = 100 volts/cm n$_0$=10^{15}/cm^3
a = 1 cm
j = CURRENT DENSITY, amp/cm^2
10^6
10^5
10^4
10^3
10^2
10
0
0.5
1.0
1.5
y-a IN CENTIMETERS
t = 0.25 μ sec
t = 0.10 μ sec
t = 0.05 μ sec
t = 0.025 μ sec
t = 0.005 μ sec

Figure 4

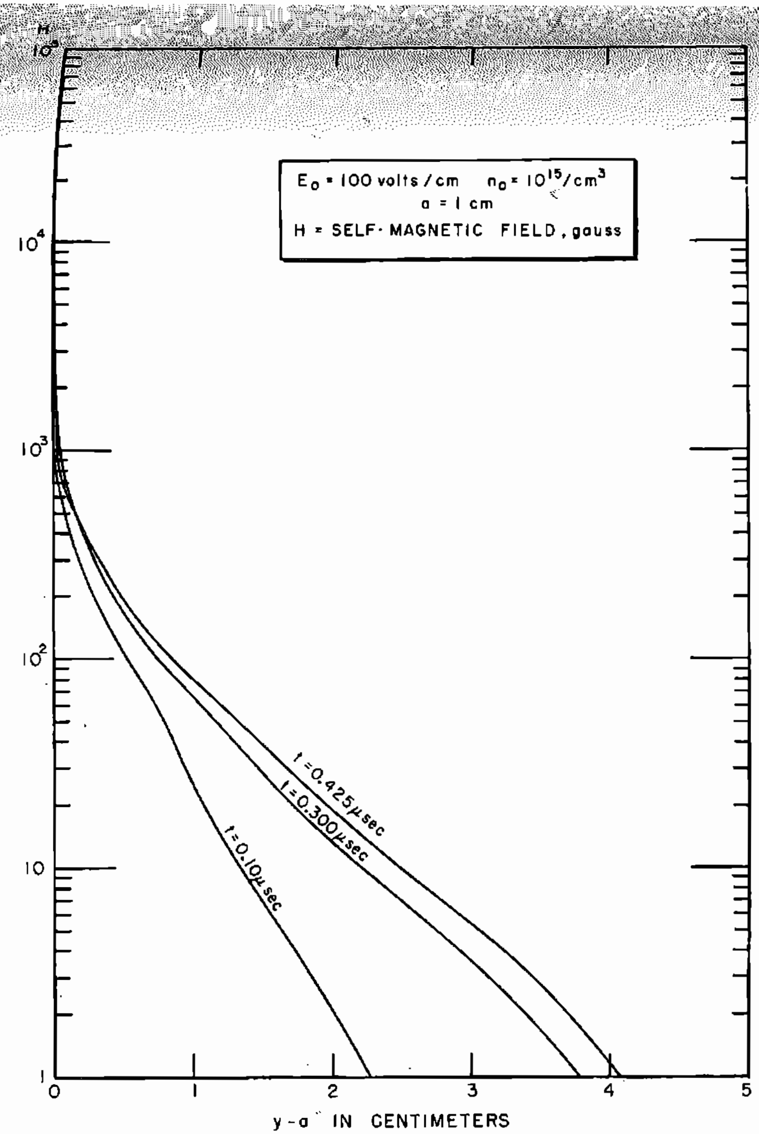


Figure 5

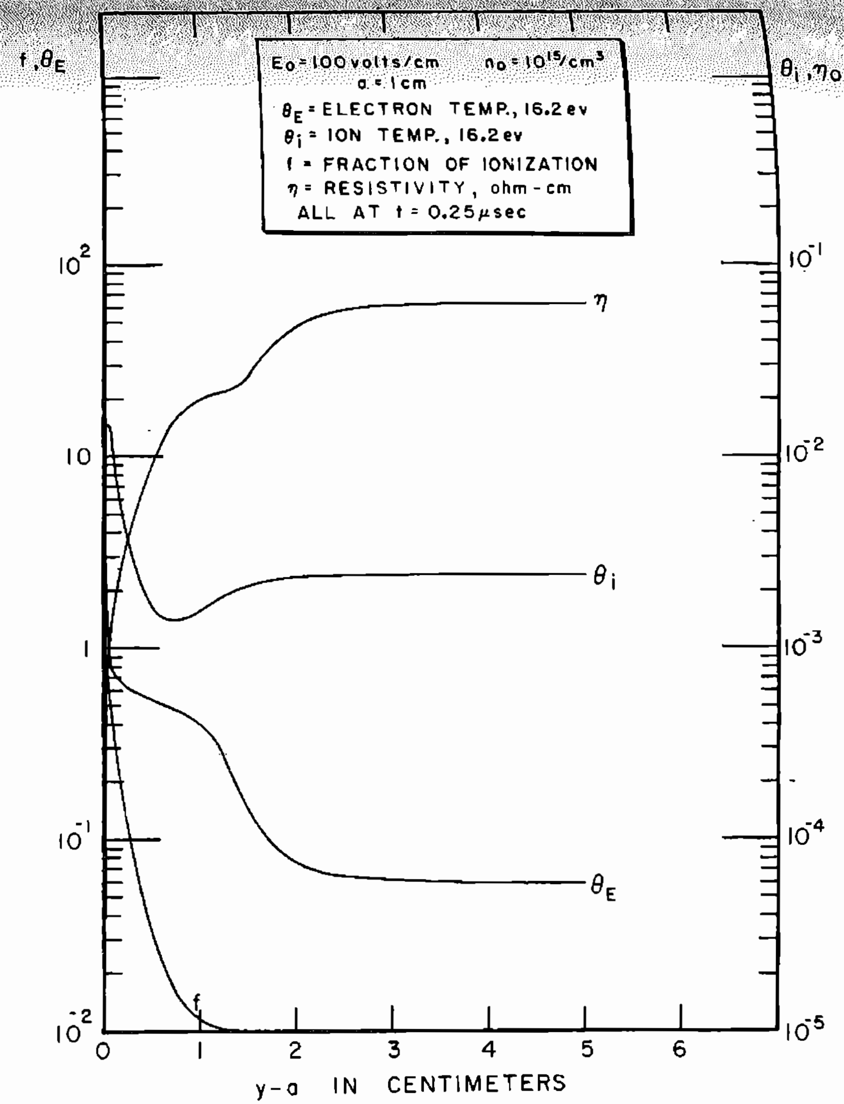


Figure 6

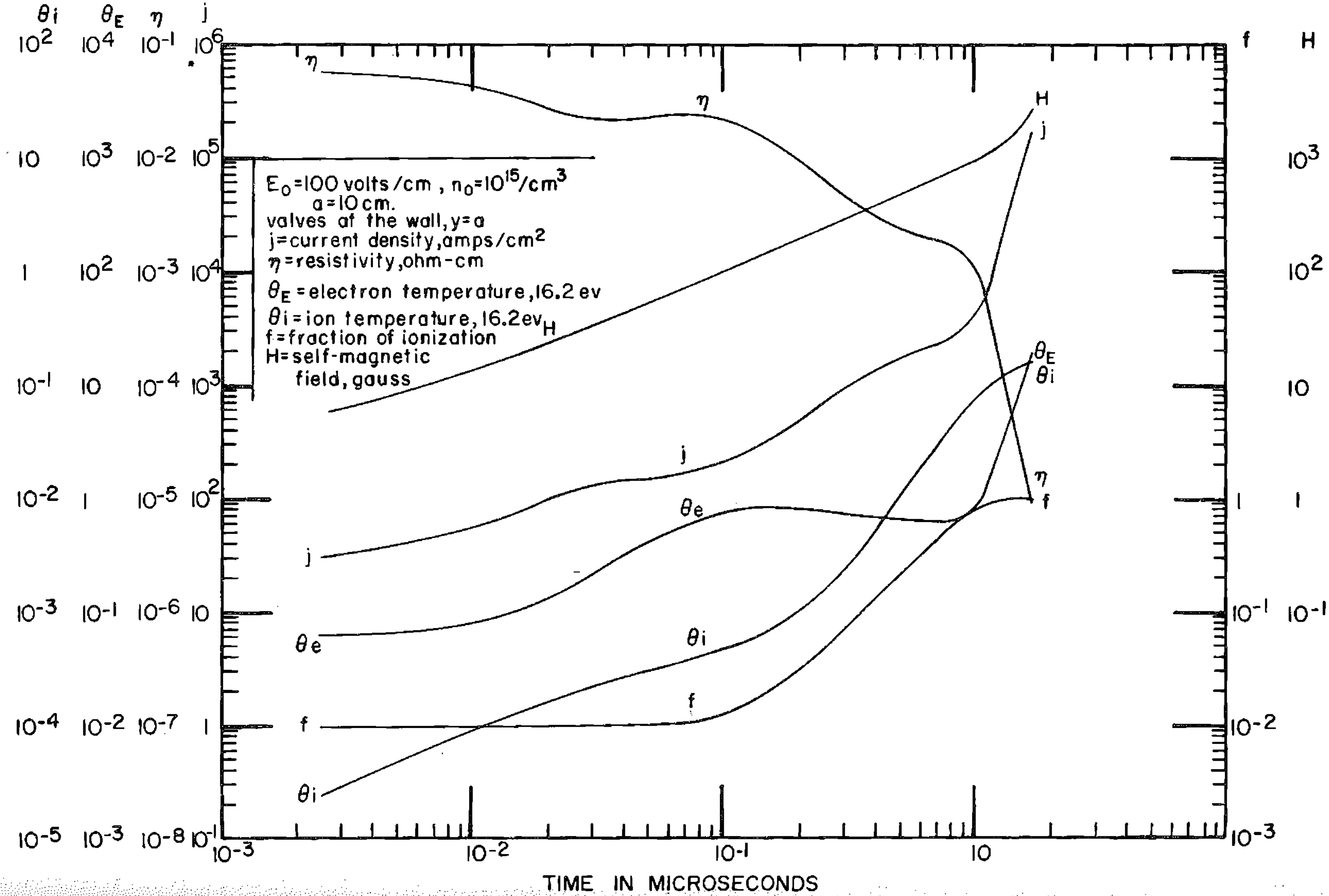

θi θE η j
E0=100 volts/cm, n0=10^15/cm^3
a=10 cm.
valves at the wall, y=a
j=current density, amps/cm^2
η=resistivity, ohm-cm
θE=electron temperature, 16.2 ev
θi=ion temperature, 16.2 ev
f=fraction of ionization
H=self-magnetic field, gauss
TIME IN MICROSECONDS
f H

Figure 7

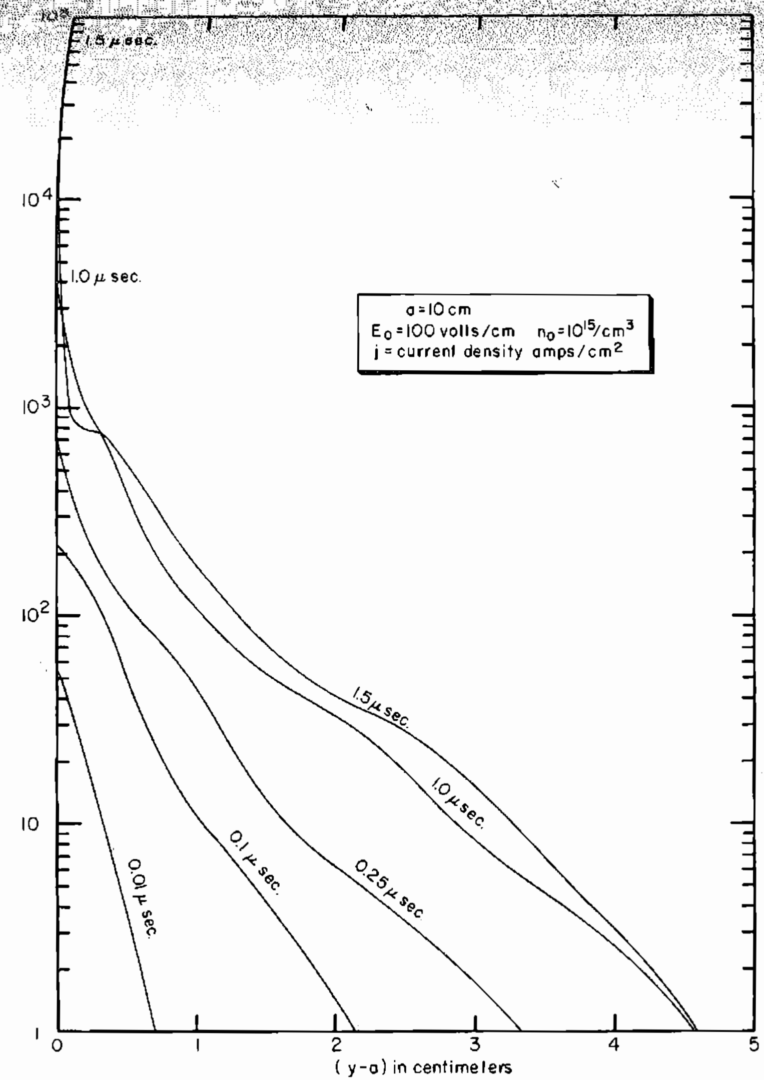


Figure 8

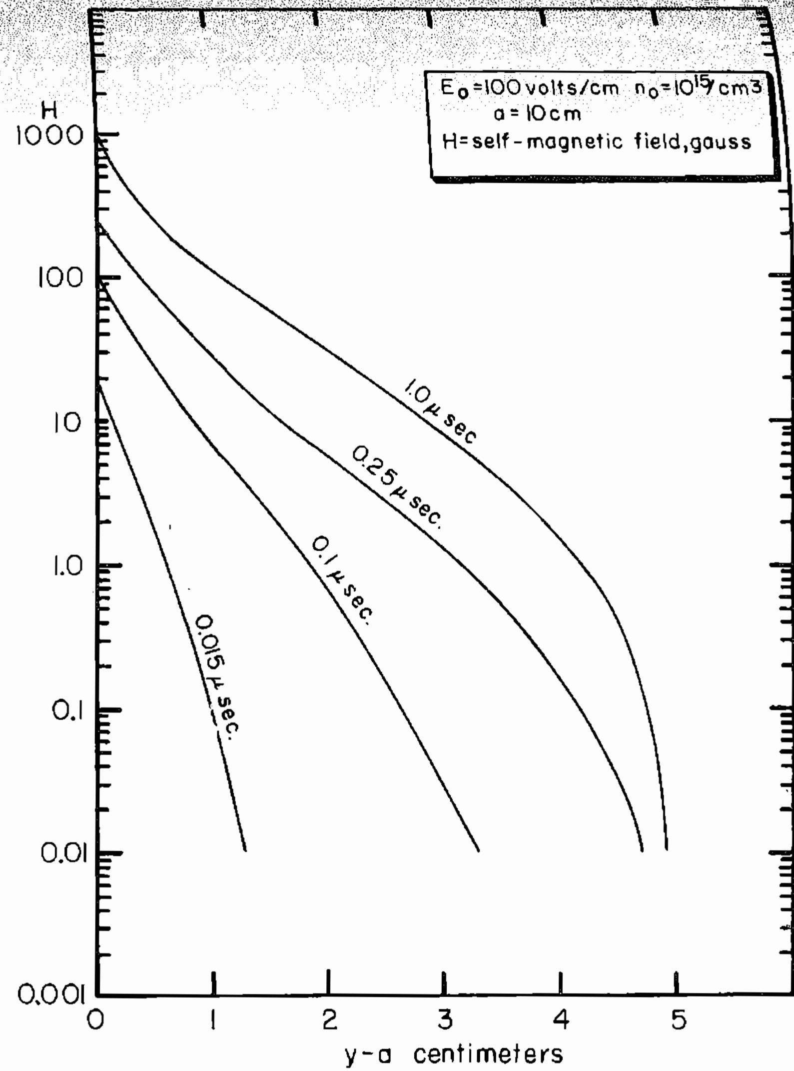


Figure 9

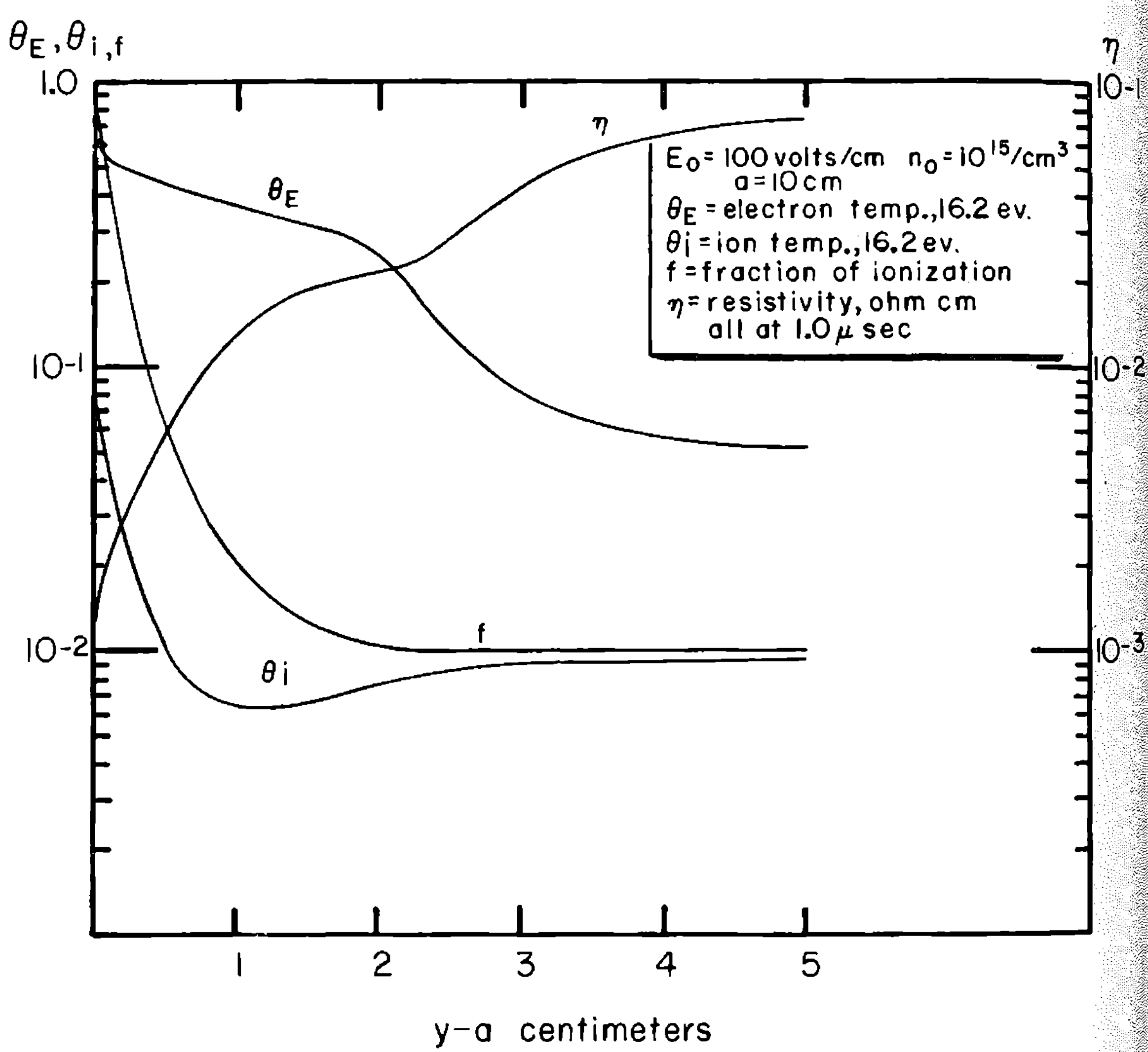
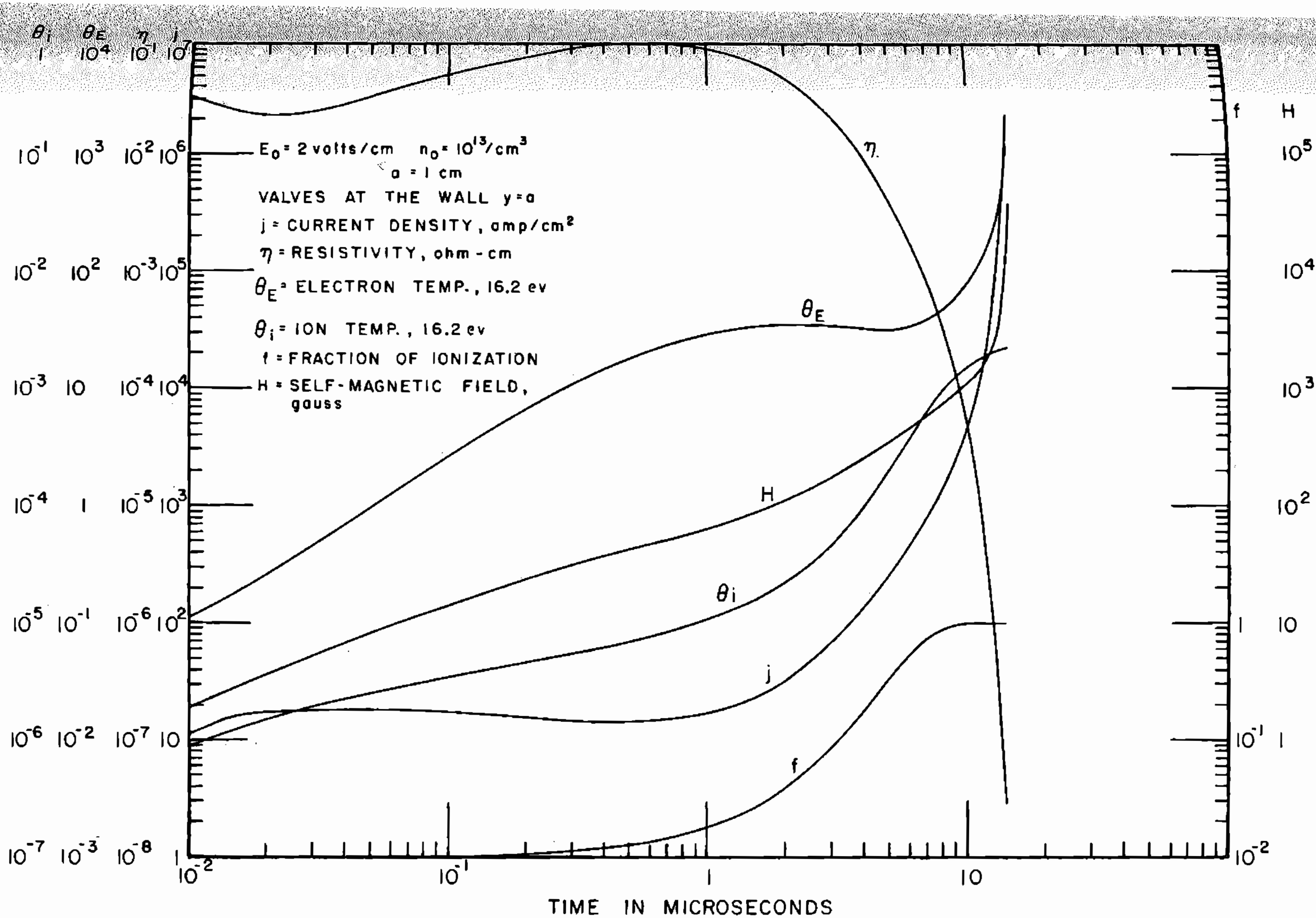

θ_E, θ_i, f
1.0
10^{-1}
10^{-2}
η
10^{-1}
10^{-2}
10^{-3}
η
θ_E
f
θ_i
E_0 = 100 volts/cm $n_0 = 10^{15}$/cm^3
a = 10 cm
θ_E = electron temp., 16.2 ev.
θ_i = ion temp., 16.2 ev.
f = fraction of ionization
η = resistivity, ohm cm
all at 1.0 μ sec
1
2
3
4
5
y−a centimeters

Figure 10



θ_i θ_E η j
E_0 = 2 volts/cm n_0 = 10^{13}/cm^3
a = 1 cm
VALVES AT THE WALL y = a
j = CURRENT DENSITY, amp/cm^2
η = RESISTIVITY, ohm-cm
θ_E = ELECTRON TEMP., 16.2 ev
θ_i = ION TEMP., 16.2 ev
f = FRACTION OF IONIZATION
H = SELF-MAGNETIC FIELD, gauss
η
θ_E
H
θ_i
j
f
f H
TIME IN MICROSECONDS

Figure 11

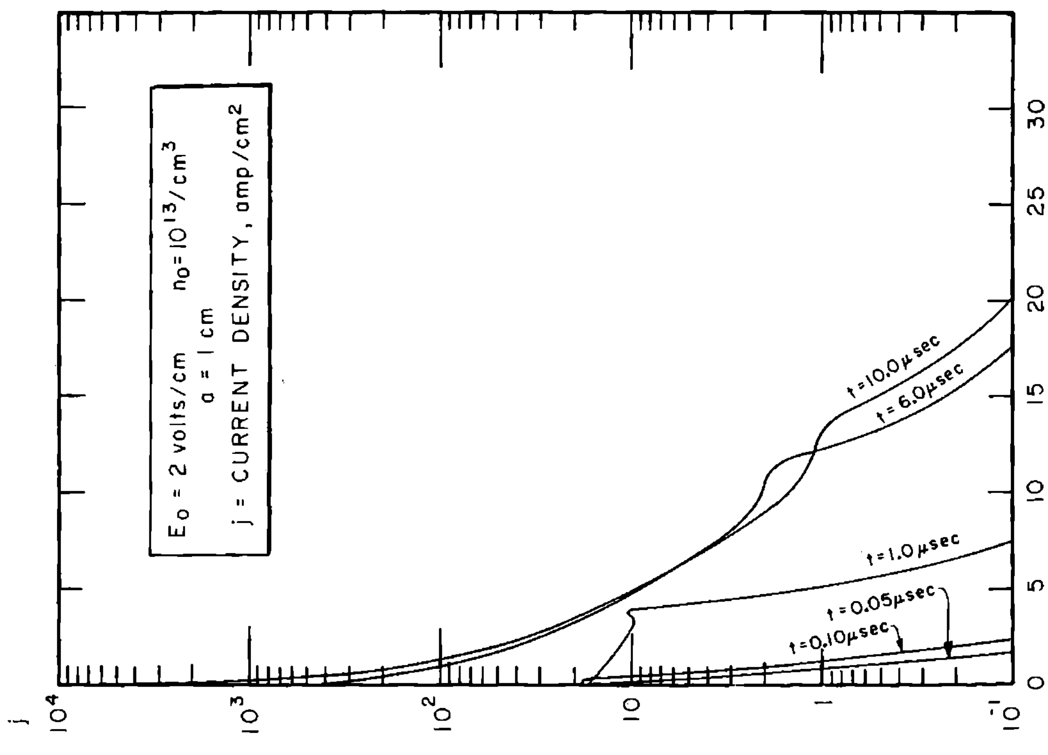


Figure 12

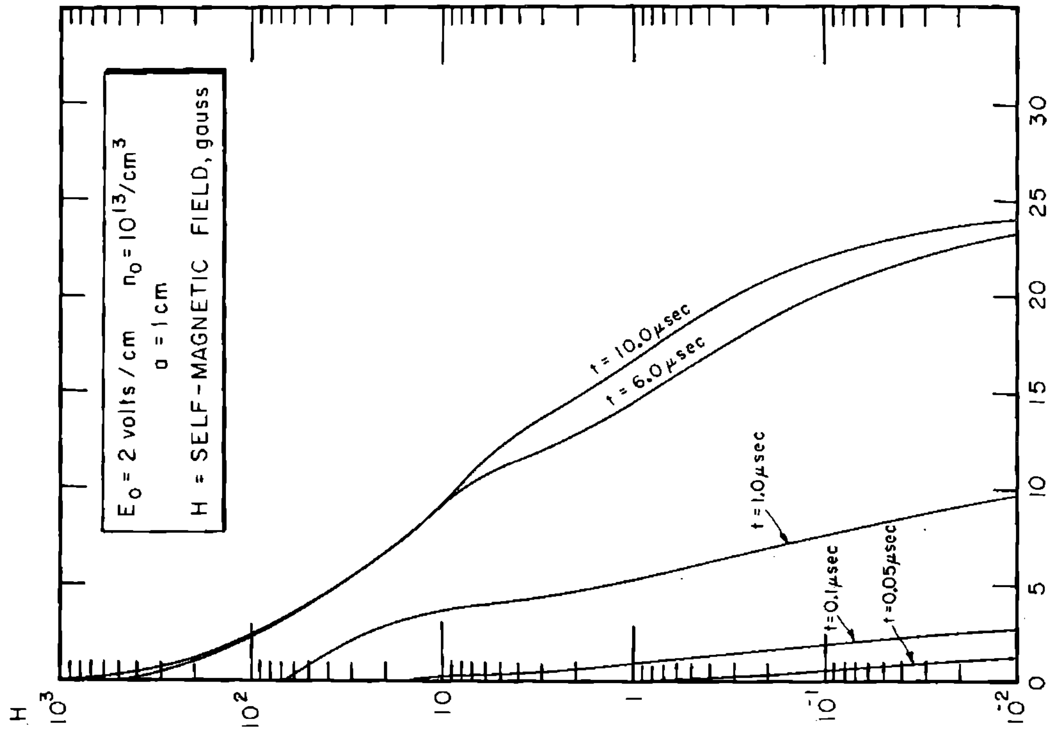


Figure 13

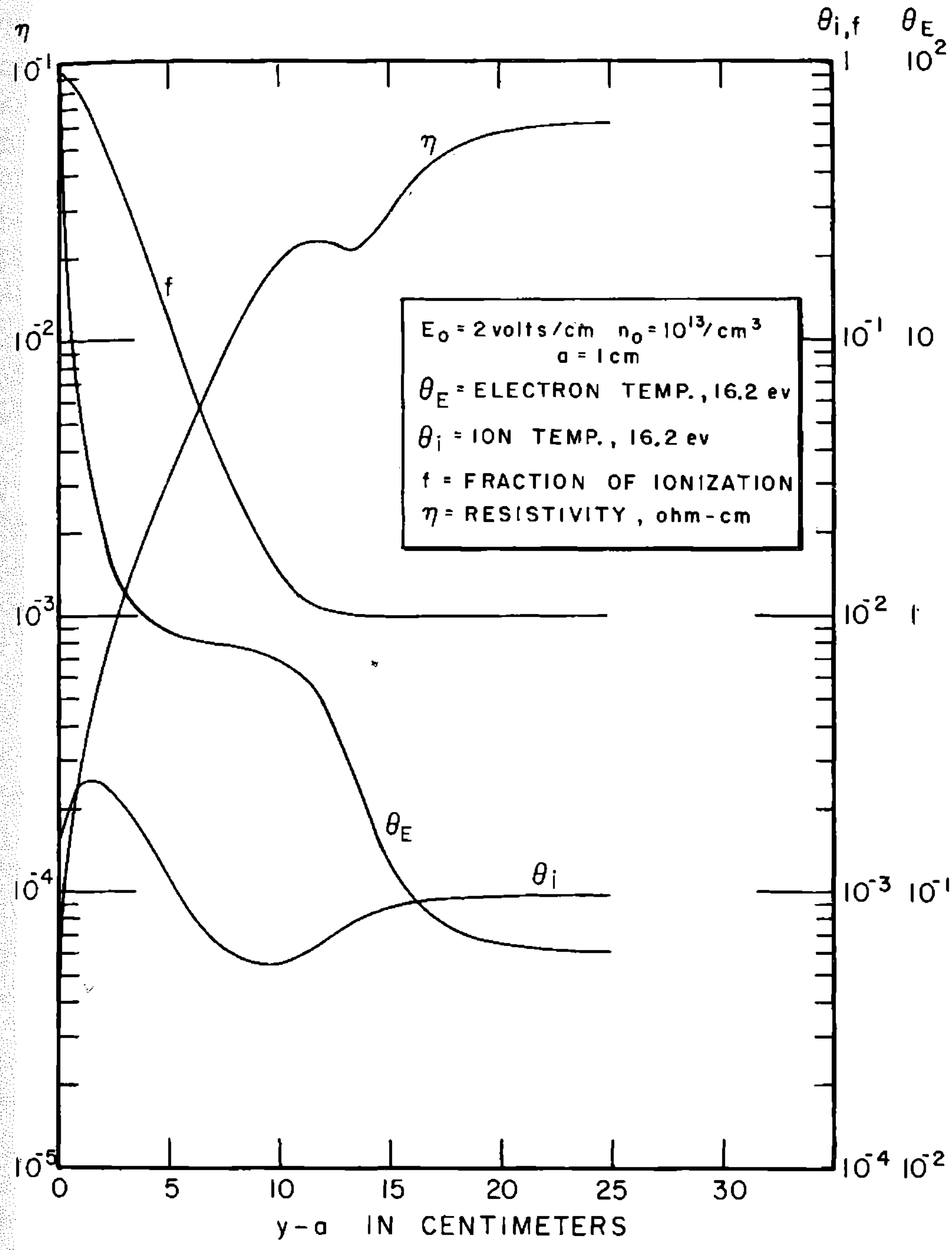
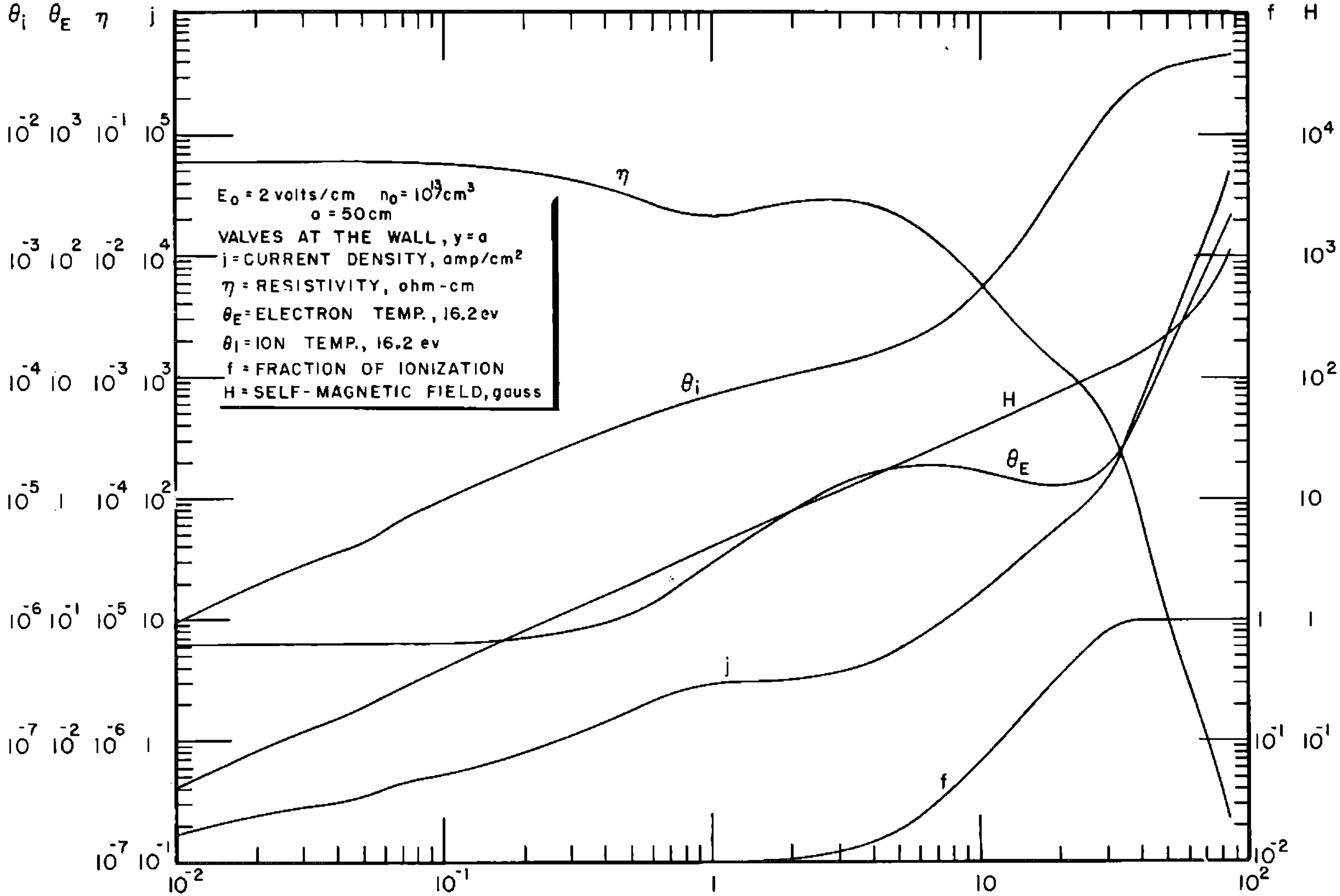

η
θi,f
θE
10⁻¹
10⁻²
10⁻³
10⁻⁴
10⁻⁵
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
10²
10
1
10⁻¹
10⁻²
f
η
θE
θi
E0 = 2 volts/cm n0 = 10¹³/cm³
a = 1 cm
θE = ELECTRON TEMP., 16.2 ev
θi = ION TEMP., 16.2 ev
f = FRACTION OF IONIZATION
η = RESISTIVITY, ohm-cm
0
5
10
15
20
25
30
y−a IN CENTIMETERS

Figure 14



θ_i θ_E η j
f H
E_o = 2 volts/cm n_o = 10^{13}/cm³
a = 50cm
VALVES AT THE WALL, y = a
j = CURRENT DENSITY, amp/cm²
η = RESISTIVITY, ohm-cm
θ_E = ELECTRON TEMP., 16.2 ev
θ_I = ION TEMP., 16.2 ev
f = FRACTION OF IONIZATION
H = SELF-MAGNETIC FIELD, gauss
TIME IN MICROSECONDS
29790-2

Figure 15

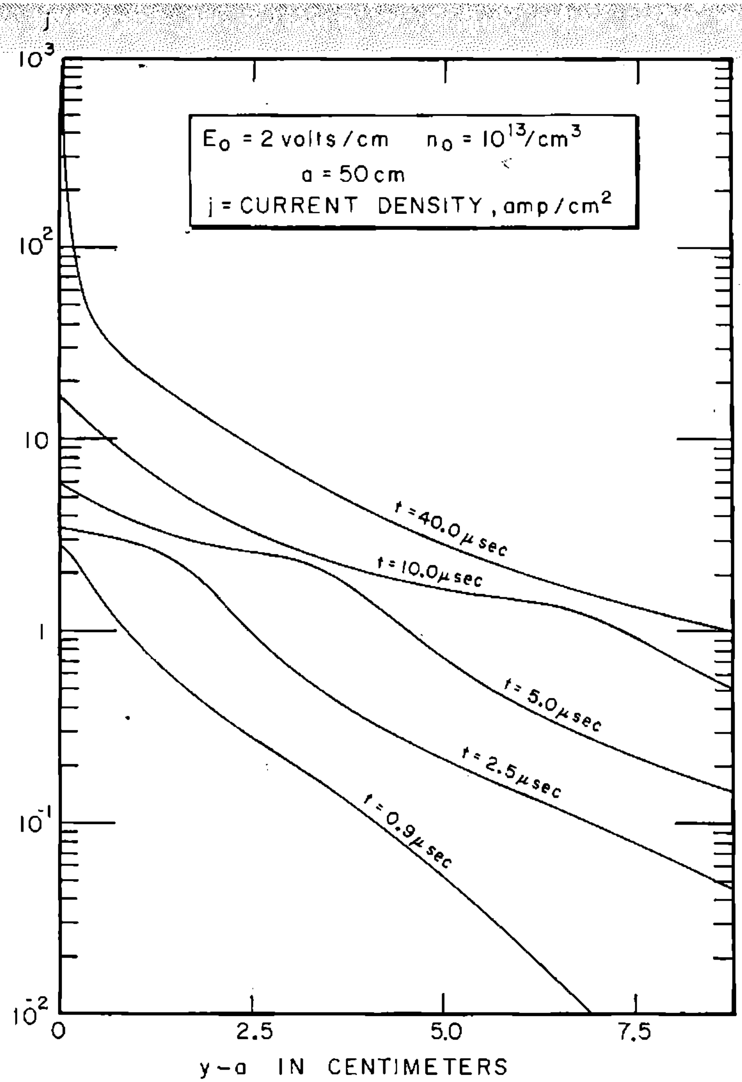


Figure 16

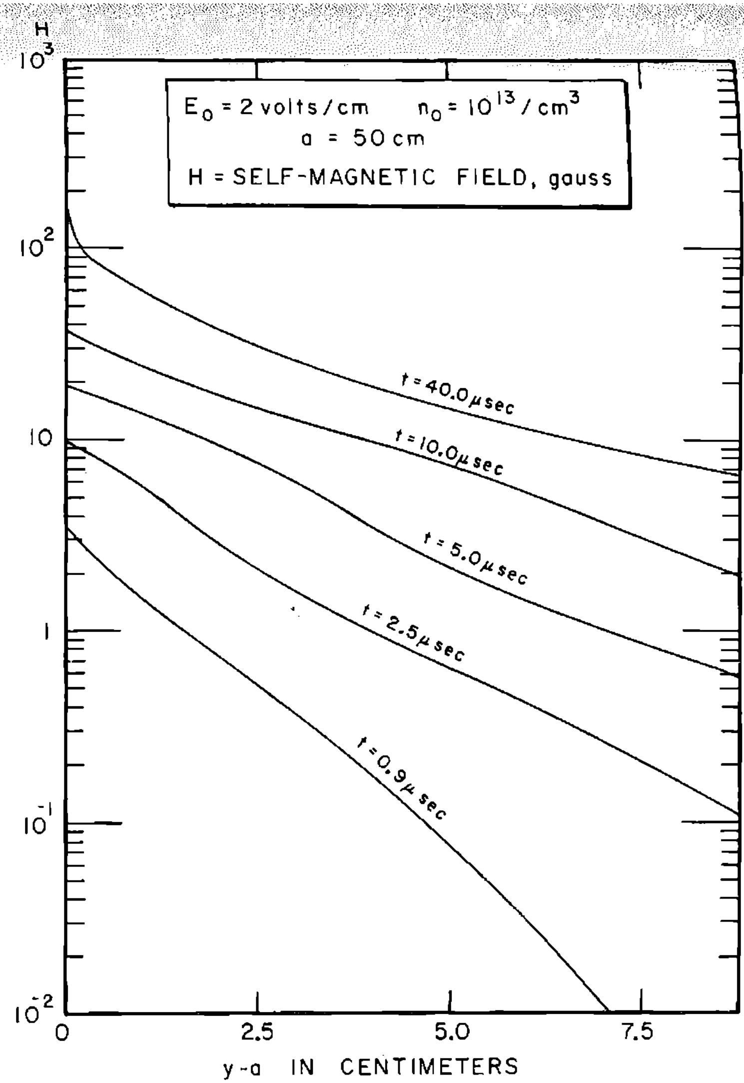


Figure 17

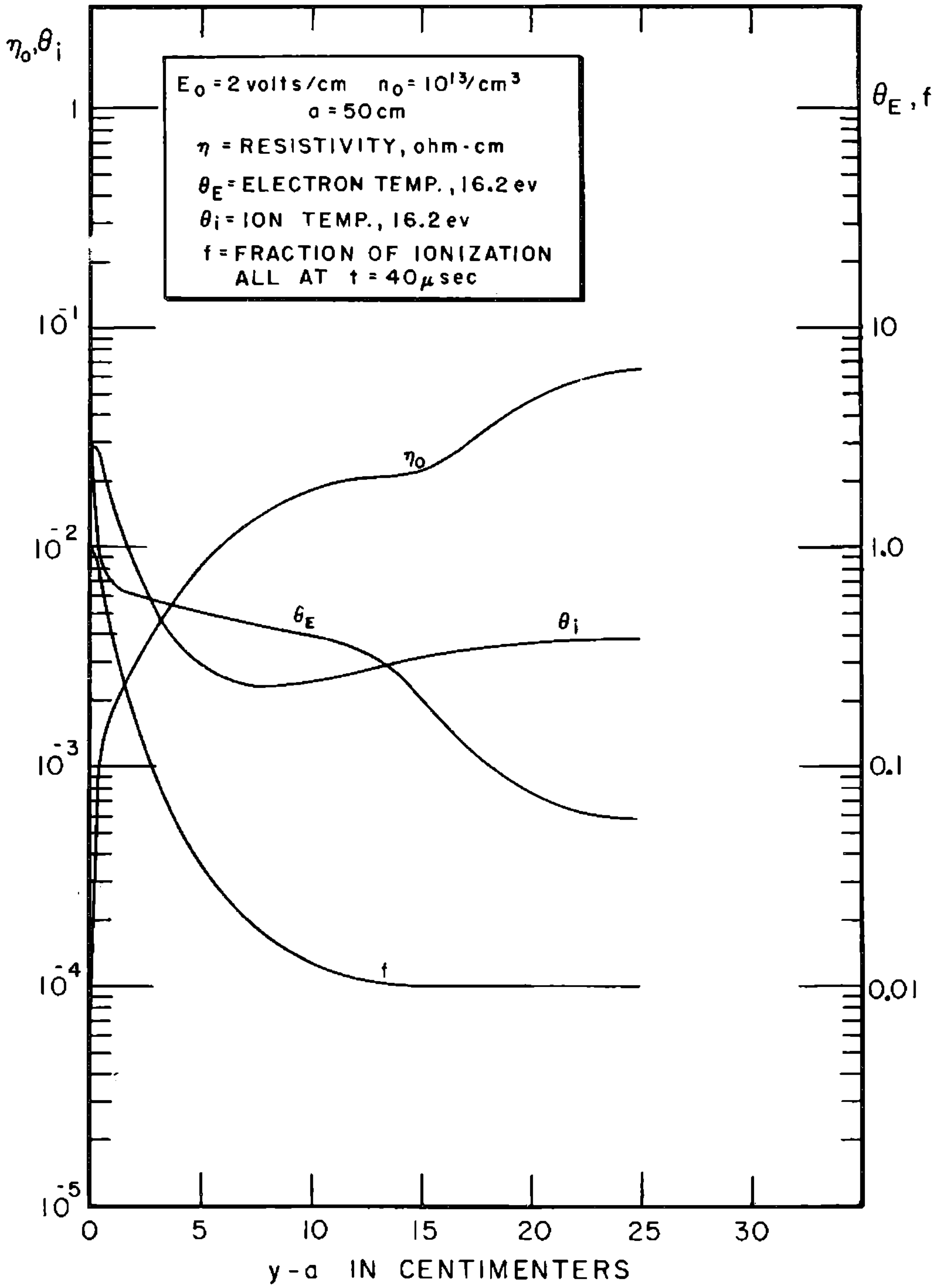

η_o, θ_i
θ_E, f
E_o = 2 volts/cm n_o = 10^{13}/cm^3
a = 50 cm
η = RESISTIVITY, ohm-cm
θ_E = ELECTRON TEMP., 16.2 ev
θ_i = ION TEMP., 16.2 ev
f = FRACTION OF IONIZATION
ALL AT t = 40μsec
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10
1.0
0.1
0.01
η_o
θ_E
θ_i
f
0
5
10
15
20
25
30
y-a IN CENTIMENTERS

Figure 18

Suppose the field in the symmetry plane containing the torus is obtained by treating the torus as a single circular current loop. This field may be approximated by the analytic expression

$$H_\theta = \frac{i}{c}\left[\frac{2}{r} + \frac{4}{R}\right]$$

The total flux due to the plasma current which passes through the summetry plane and enclosed by the torus is

$$\phi = 2\pi\mu\int_R^{r_o} H_\theta \quad (R - r)\,d\,r,$$

and using the above expression for H_θ gives

$$\phi = 4\pi\mu R\,\frac{i}{c}\left[\alpha^2 - \alpha - \ln\alpha\right],$$

where $\alpha = \frac{r_o}{R}.$

Using the analytic expression for H_θ gives a total enclosed flux which agrees to within 1/2 o/$_o$ of the numerical result. Now in the center of the torus, i.e., at $r = R$, assume that there is a changing externally applied magnetic field which gives a constant electric field, E_o, at the plasma.

$$E_o = -\frac{1}{c}\,\frac{1}{2\pi R}\,\frac{\partial\phi_o}{\partial t}$$

Hence the electric field at the boundary of the plasma is

$$E_B = E_o - \frac{2\mu}{c^2}\left[\alpha^2 - \alpha - \ln\alpha\right]\frac{\partial i}{\partial t}.$$

In our one dimensional model we have from Eqs. (6) and (8)

$$E_B = E_o - \frac{4\pi\mu}{c^2}\,a\,\frac{\partial i'}{\partial t}$$

where $i' = \frac{i}{2\pi r_o}$ for the torus, so

$$\frac{a}{r_o} = \alpha^2 - \alpha - \ln\alpha \tag{30}$$

Eq. (30) enables us to calculate an effective value of a in our one-dimensional model for a particular torus.

Consider a small torus where $r_o = 5$ cm and $\frac{r_o}{R}$ about $\frac{1}{8}$ then $\frac{a}{r_o}$ is about 2. For the case $a = 10$ cm. we present a case with $E_o = 100$ volts/cm. and $n_o = 10^{15}/\text{cm}^3$ in Figs. (7) - (10).

Now consider a large torus where $r_o = 50$ cm. and $R = 160$ cm. then $\frac{a}{r_o}$ is about 1. For the case $a = 50$ cm. we present a case with $E_o = 2$ volts/cm. and $n_o = 10^{13}/\text{cm}^3$ in figs. (15) - (18).

CIRCUIT DYNAMICS OF THE PINCH*

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Abstract

Instead of analyzing in detail a portion of a hydromagnetic pinch apparatus and replacing the remainder by a boundary condition, the entire pinch apparatus is treated as a single dynamical system. A circuit equation and a mechanical equation, coupled together, result. These equations describe the dynamical development of the pinch and exhibit explicitly its dependence on the physical parameters (electrical and mechanical) of the system. As examples, the equations have been used to analyze the snow-plow model and the adiabatic pinch, yielding curves that show the geometrical development of the pinch in time, as well as the distribution of mechanical and magnetic energies at any stage. Analogous analyses may be made for other physical quantities of interest, and can be used to adjust the parameters of the system so as to optimize specific pinch characteristics.

Dr. B. A. Lippmann: I should like to talk about some work that John Killeen and I did a couple of summers ago. The problem is neither particularly subtle nor particularly difficult. In some respects it is trivial, and in a minor way it is even useful.

What we did was to analyze the behavior of a pinch tube, including the reaction back on the source. That is, instead of considering the pinch tube in detail, and representing the rest of the system by a boundary condition, we considered the entire apparatus as a single dynamical system.

There were several reasons for doing this. One reason was that it could be done. Since everything about the external circuitry is known, there seems to be little reason to leave it out of the analysis. We also noticed that one can calculate everything in detail if the geometry is simple enough; for example, one can derive expressions for the rate at which energy is put into the magnetic field, the rate at which work is done on the plasma, etc.

The variation of these quantities as the pinch develops in time can be calculated, and we felt that it might be quite useful to have this information available. It is the fundamental prerequisite for a quantitative understanding

*This work was performed under the auspices of the U. S. Atomic Energy Commission.

of the pinch, its diagnostics, and its design, and, in fact, that is what the information gained has been used for at Livermore.

If we consider only the simplest case, a pinch apparatus is equivalent to a condenser and an inductance connected in series. The circuit equation is:

$$\frac{Q}{C} + \frac{d}{dt}(LI) = 0$$

where Q is the charge on the condenser bank, C is its capacitance, and L is the inductance of the pinch tube.

We also have a mechanical equation:

$$W = \frac{1}{2} I^2 \frac{dL}{dt},$$

where W is the rate at which work is done on the plasma.

The inductance is known if, as we shall assume, the pinch tube geometry is co-axial. However, we need a mechanical model before we can compute the W to be used in the second equation. We have analyzed two models: the snow plow model and the model of an adiabatic pinch. The snow plow model has meaning for the ordinary linear or toroidal pinch tube, while the adiabatic model bears a strong resemblance to Zeta.

For the adiabatic case, the mechanical equation integrates immediately because it can be replaced by the adiabatic condition:

$$PV^{\gamma} = \text{constant}.$$

But P is proportional to I^2/R^2, and the volume in the case of a co-axial pinch cylinder is proportional to R^2. So we find:

$$IR^{\gamma-1} = \text{constant}.$$

This information is put back into the circuit equation, which is then integrated. John Killeen will show you some of the curves that result.

There is just one other matter that I think ought to be mentioned and that concerns the validity of using circuit equations in problems of this type. Suppose the geometry was much more complicated and suppose the mechanical aspects of the equations offered no difficulties, could the electrical features really be analyzed using circuit equations? The answer is "yes," in the following sense.

The circuit equations always work because they are a way of representing a solution of the Maxwell equations. However, one will not always encounter a circuit as simple as the series L, C combination we have considered here. Our circuit was simple because dissipation, whether in the form of ohmic losses or radiation losses, was negligible. In addition, we have assumed that the solution of the Maxwell equations could be described in terms of definite regions where the electric and magnetic energies were concentrated (C and L), coupled together by only dominant mode interactions.

In more general situations, we would expect to find radiation and ohmic losses, as well as multi-mode interactions. The circuit equation would then be replaced by a set of circuit equations, which, although complicated, could be written down using the known techniques of the theory of guided waves.

I will now ask John to show you the curves.

DR. KILLEEN: In the first slide we show the equations to be solved in the snowplow model. The circuit equation becomes

$$\frac{d}{d\tau}\left[(b - \ln \eta)\frac{dq}{d\tau}\right] = -q \quad ,$$

and the mechanical equation becomes

$$\frac{d}{d\tau}\left[(1 - \eta^2)\frac{d\eta}{d\tau}\right] = -\frac{1}{\eta}\left(\frac{dq}{d\tau}\right)^2 \quad ,$$

where

$$\eta = \frac{r}{Rw} \; ; \quad q = \frac{Q}{10\, R_w^{\,2}\sqrt{\pi \rho_o}}$$

$$\tau = \frac{t}{\sqrt{2 \times 10^{-9}\, \ell_p\, c}} \; ; \quad b = \ln \frac{Rs}{Rw} + \frac{Le}{2 \times 10^{-9}\, \ell_p}$$

The results shown for this model in the next few slides are for a device under construction at Livermore. It is a linear pinch with tube length, ℓ_p = 30 cm. The radius of the tube, Rw = 15 cm, and the shell radius, Rs = 16.5 cm. The external inductance, Le, is 10^{-7} henries. The charge on the bank, Q = 5.61 coulombs, and the capacitance, c = 18.7 x 10^{-6} farads. These characteristics give the following values for the parameters

$$q(0) = 10.6$$

$$b = 1.86$$

The results presented in the slides are for q(0) = 5, 10, 15 and b = 1, 2. In the first slide (fig. 1) we have $\eta(\tau)$, which gives the position of the imploding current sheath. In the next slide (fig. 2) we have $\eta(\tau)$ which gives the velocity of the sheath. In the next slide (fig. 3) we have the current.

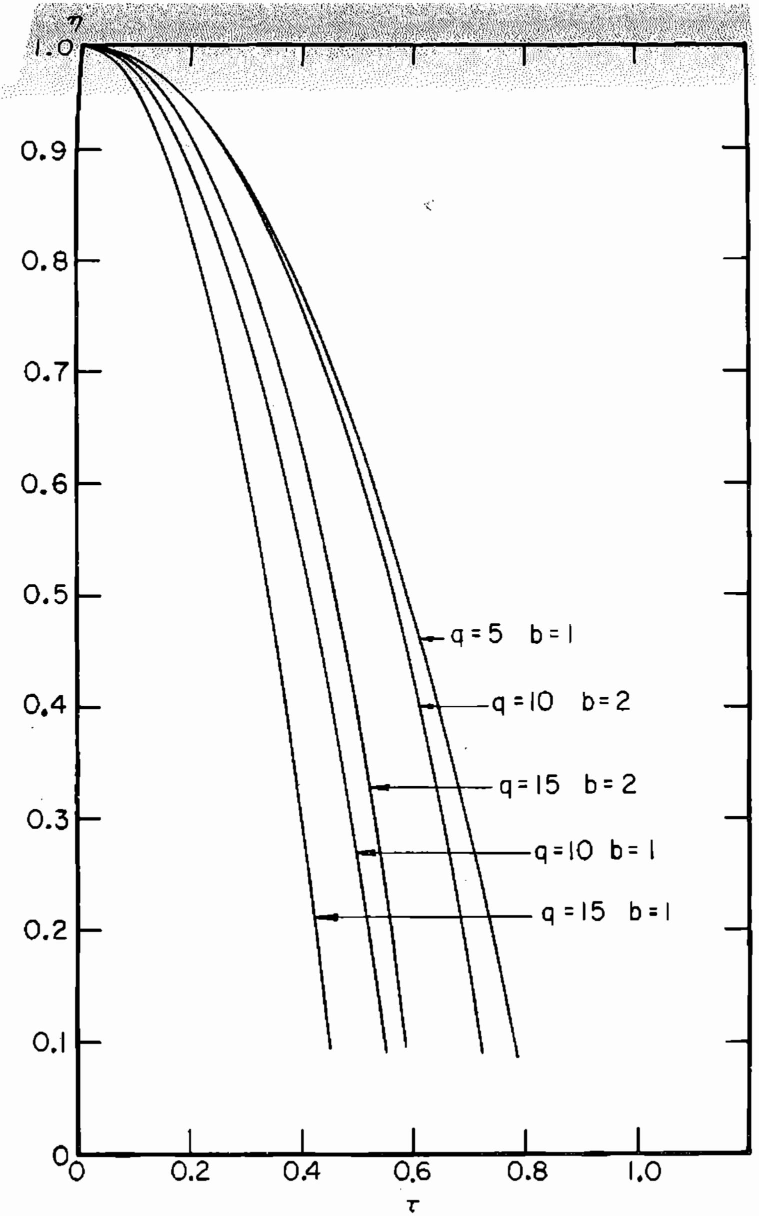

Figure 1

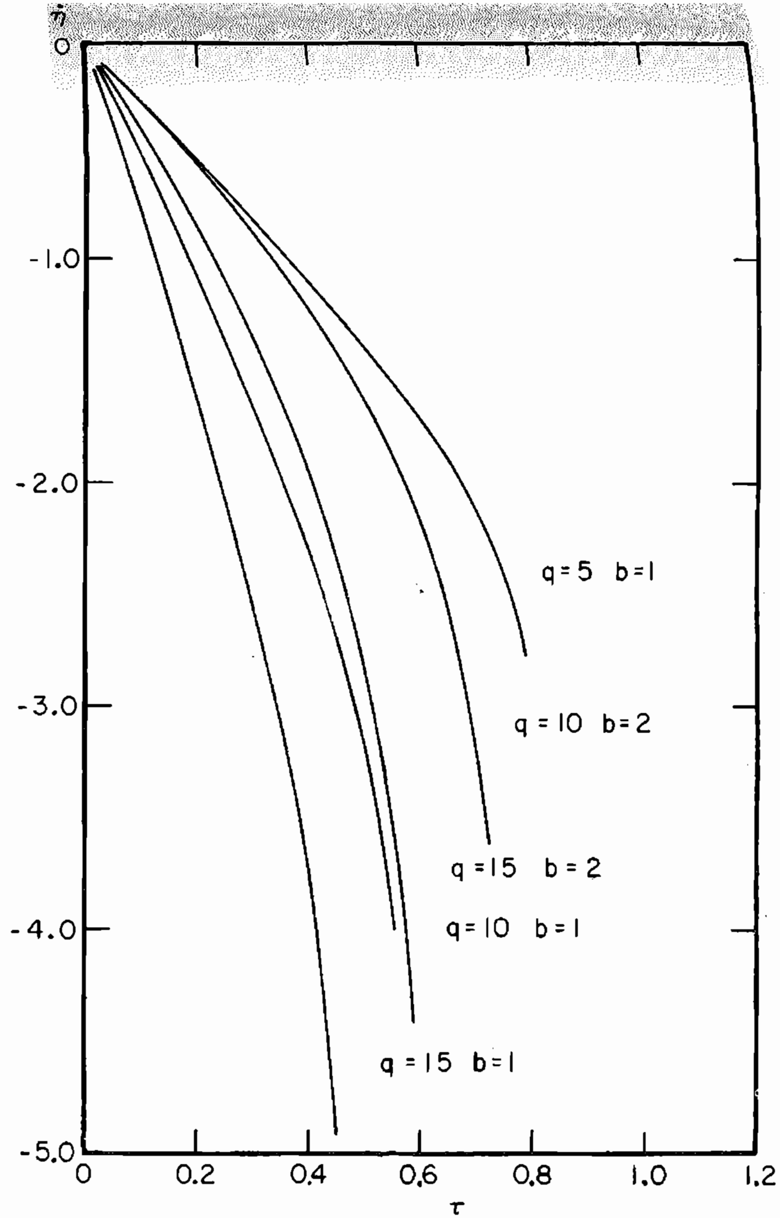

Figure 2

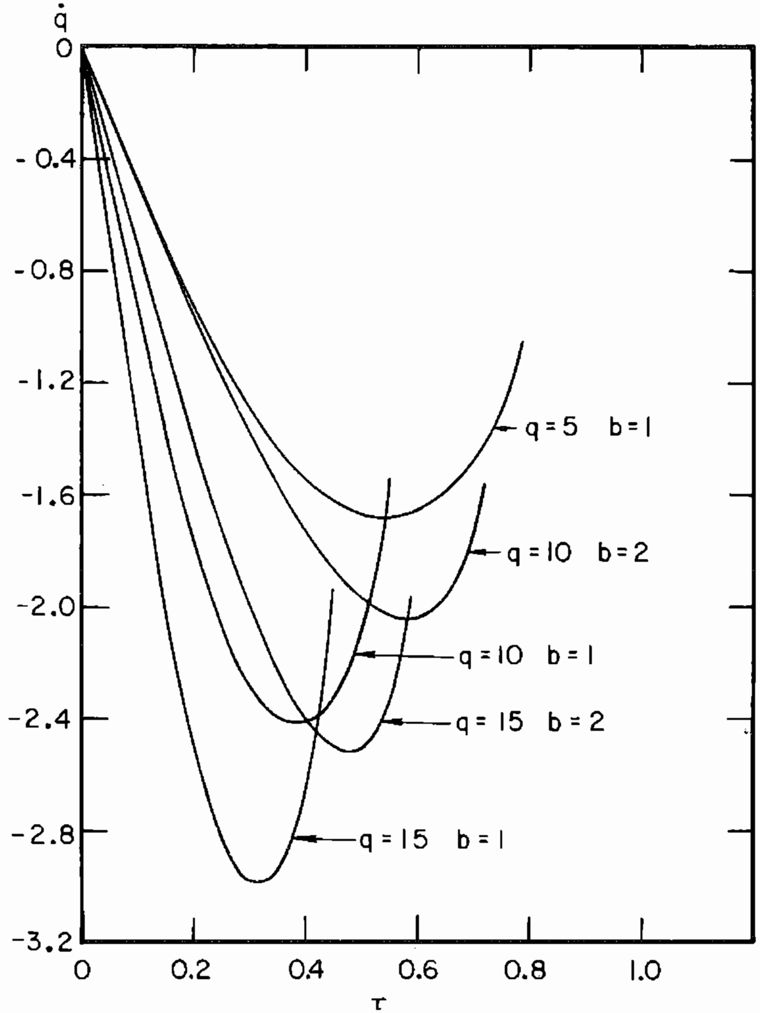

Figure 3

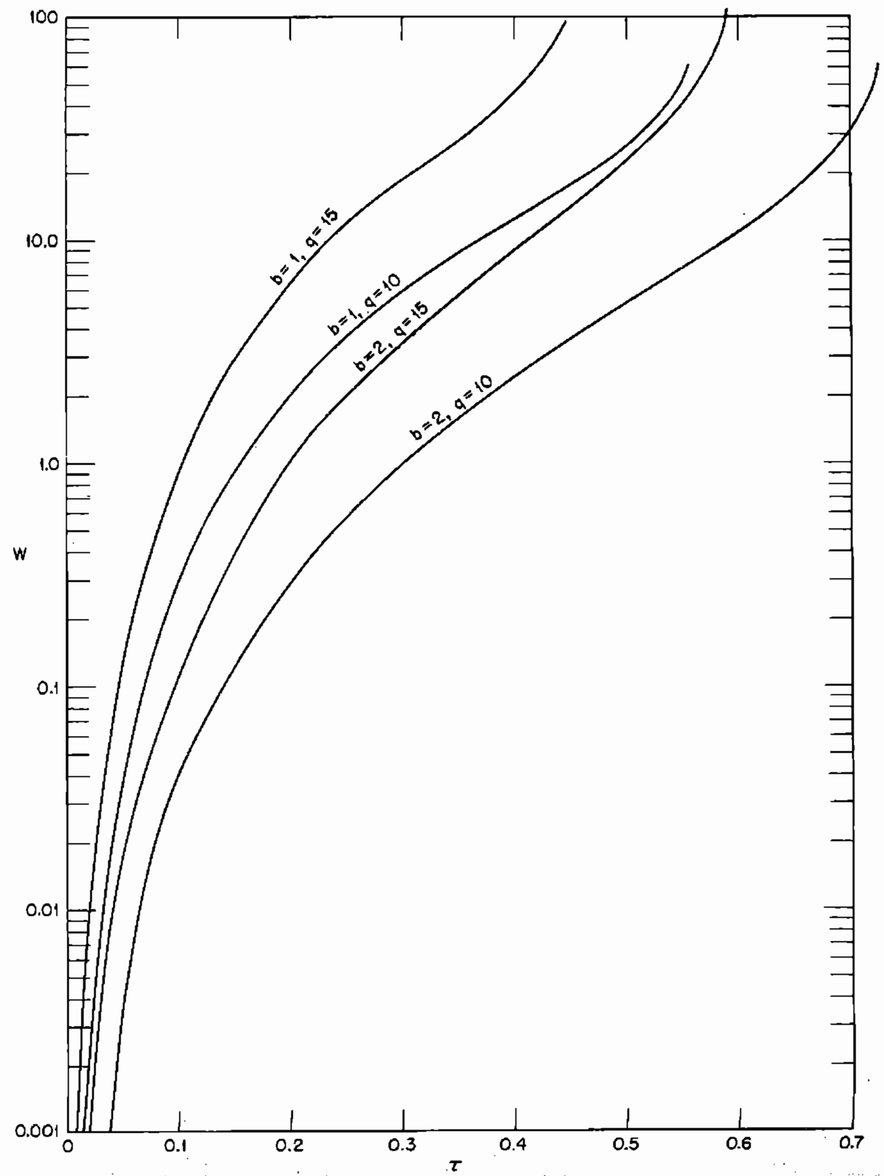

Figure 4

The energy balance equation is

$$\psi + W + \frac{dU_m}{dt} = 0$$

where

$$W = A\left[\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2 \frac{d}{d\tau}(b - \ln \eta)\right]$$

$$\frac{dU_m}{dt} = A \frac{d}{d\tau}\left[\frac{1}{2}(b - \ln \eta)\left(\frac{d\dot{q}}{d\tau}\right)^2\right]$$

$$\psi = A \frac{d}{d\tau}\left(\frac{1}{2} q^2\right)$$

where

$$A = \frac{100 R_W^4 \pi \rho_o}{\sqrt{2 \times 10^{-9} \ell_p C^3}}$$

In the next three slides we show these quantities for the cases considered.

The equations to be solved in the adiabatic model are

$$\frac{d}{d\tau}\left[(b - \ln \eta)\frac{d\bar{q}}{d\tau}\right] = -\bar{q}$$

$$\frac{d\bar{q}}{d\tau}\eta = -1$$

where

$$\bar{q} = \frac{q}{\left(\frac{dq}{d\tau}\right)_o} = \frac{Q}{\sqrt{2 \times 10^{-9} \ell_p C}\, I_o},$$

and η, τ, b are defined as before. We can solve the above equations for values of the parameters which correspond to Zeta.

Let R_s = 50 cm.

R_W = 53.5 cm.

L_e = 3.3×10^{-6} henries

ℓ_p = 1160 cm.

C = 0.13 farads

Q = 360 coulombs (25 kv)

I_o = 50×10^3 amps.

With these values b = 1.5 and q (0) = 13. In the next four slides (figs. 7 -10) are shown the results for b = 1.5, 3.0 and q(0) = 5, 8, 13, 20. The case q(0) = 13 corresponds to a voltage at the bank of 25 kv. We should note that the current

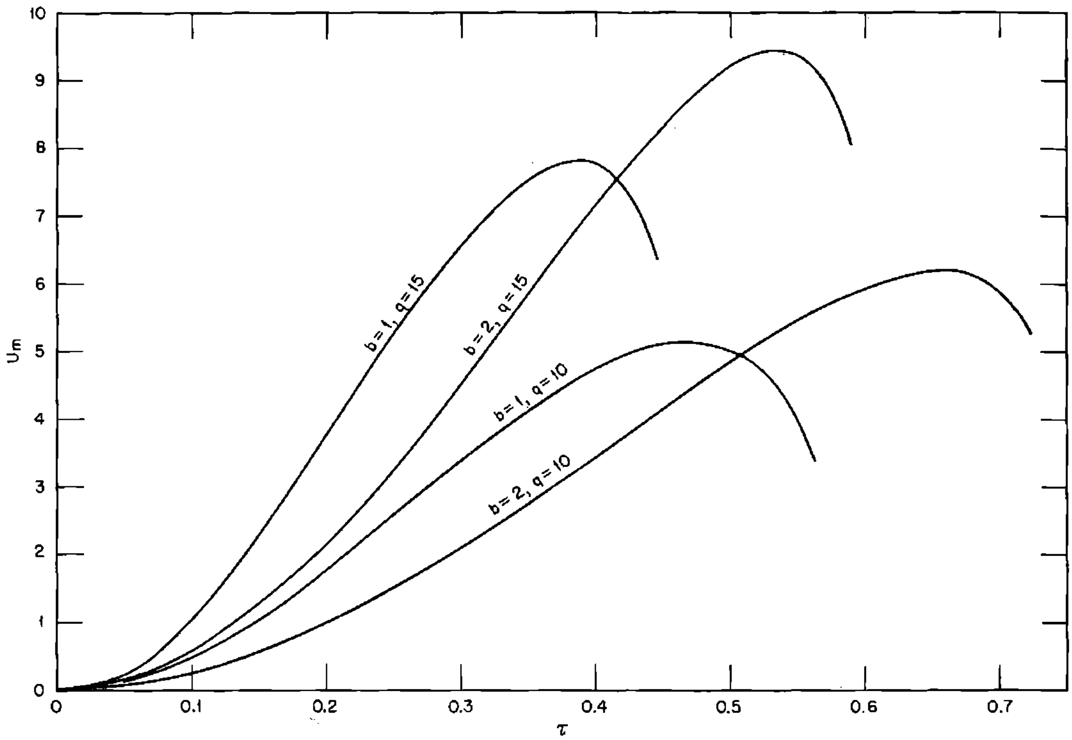

Figure 5

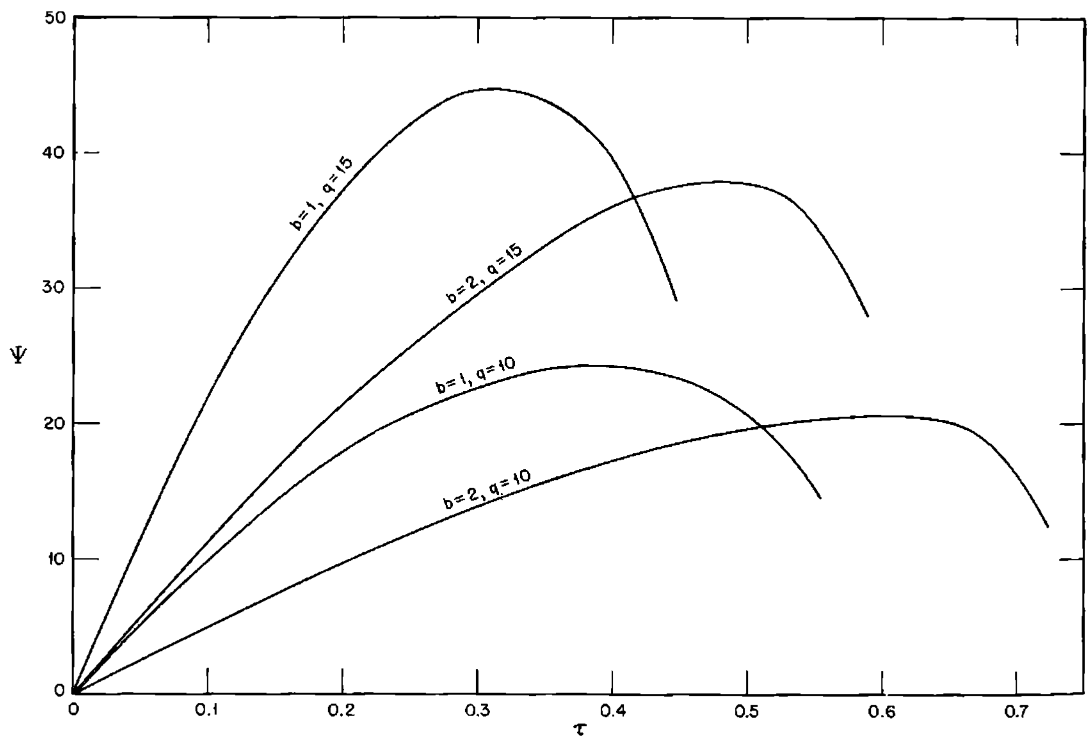

Figure 6

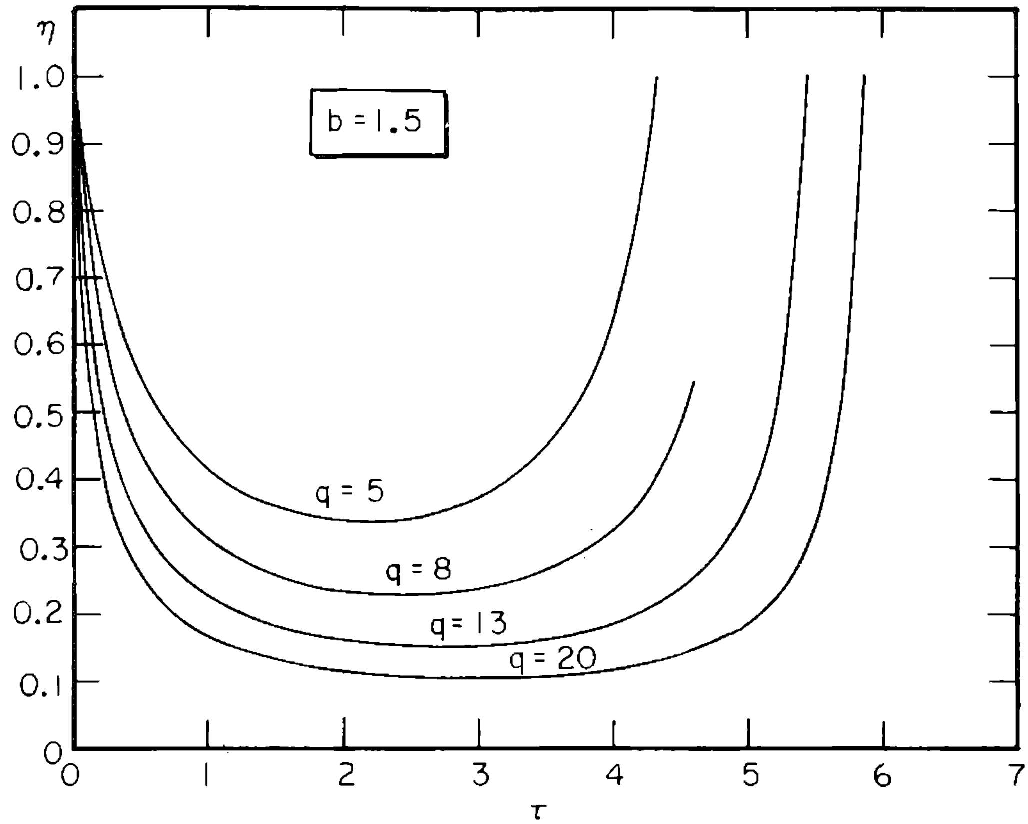

Figure 7

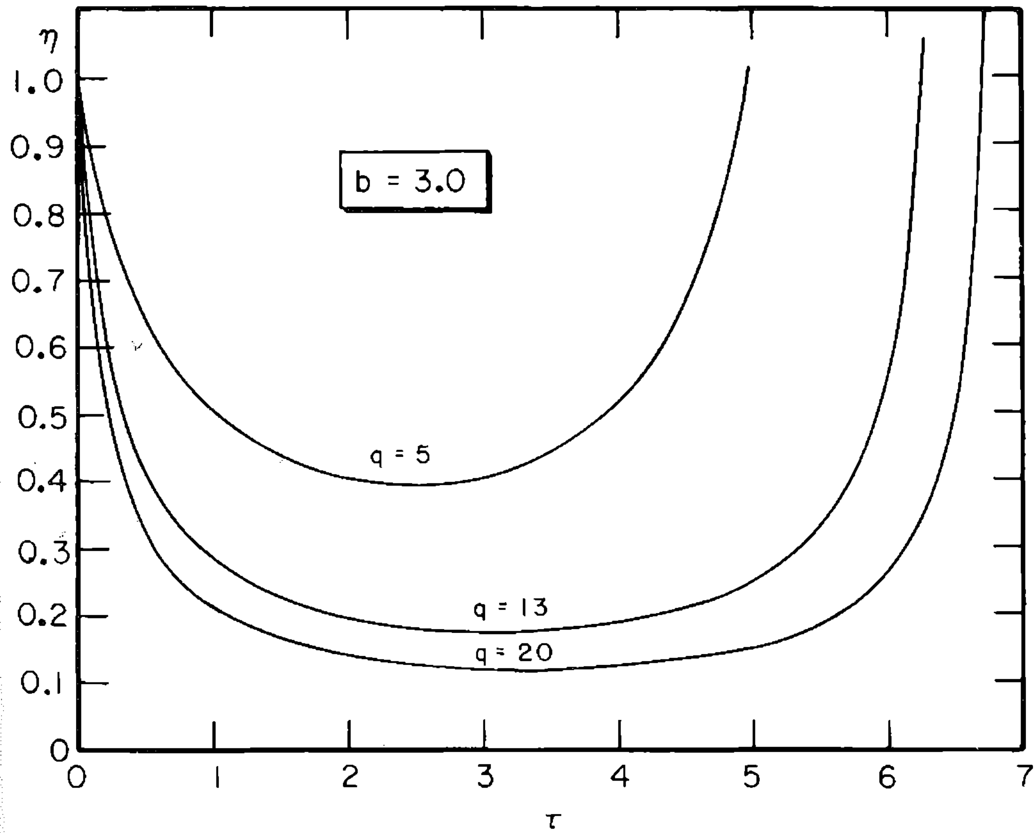

Figure 8

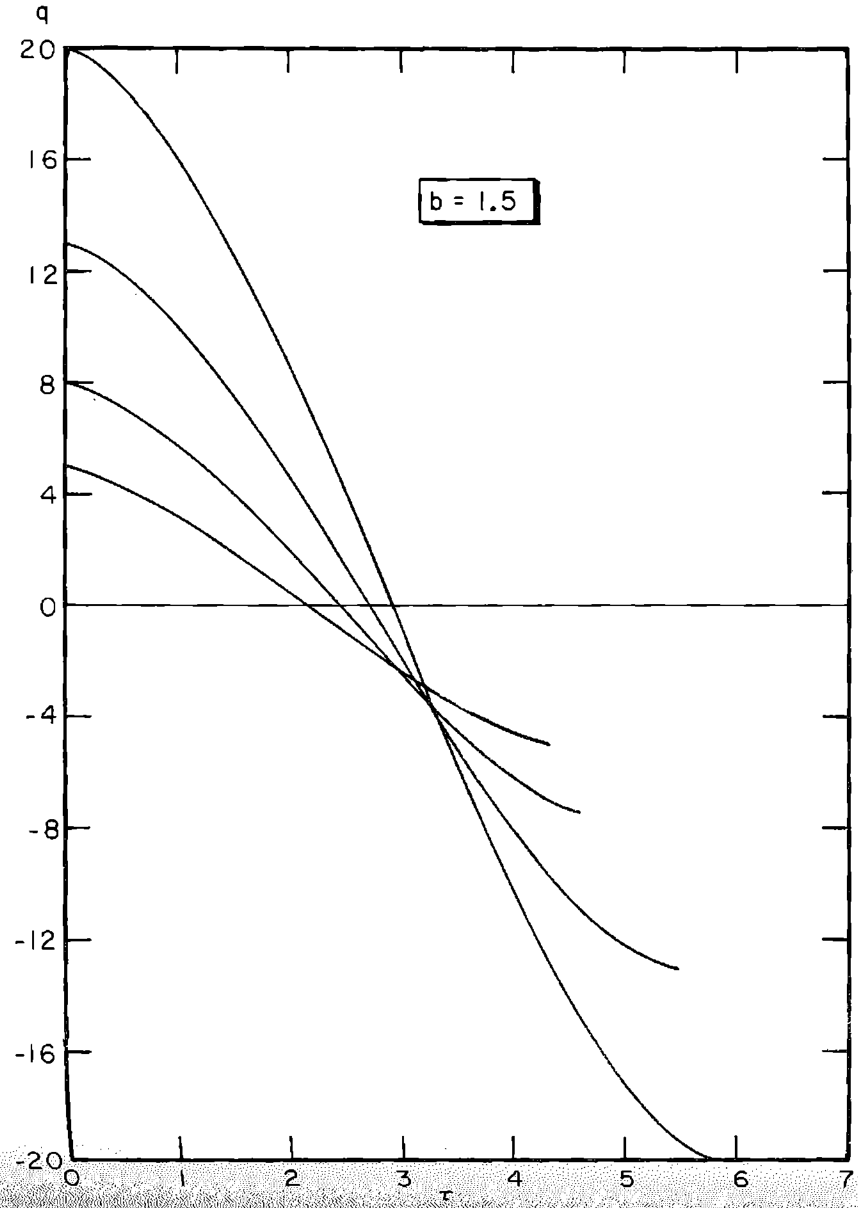

Figure 9

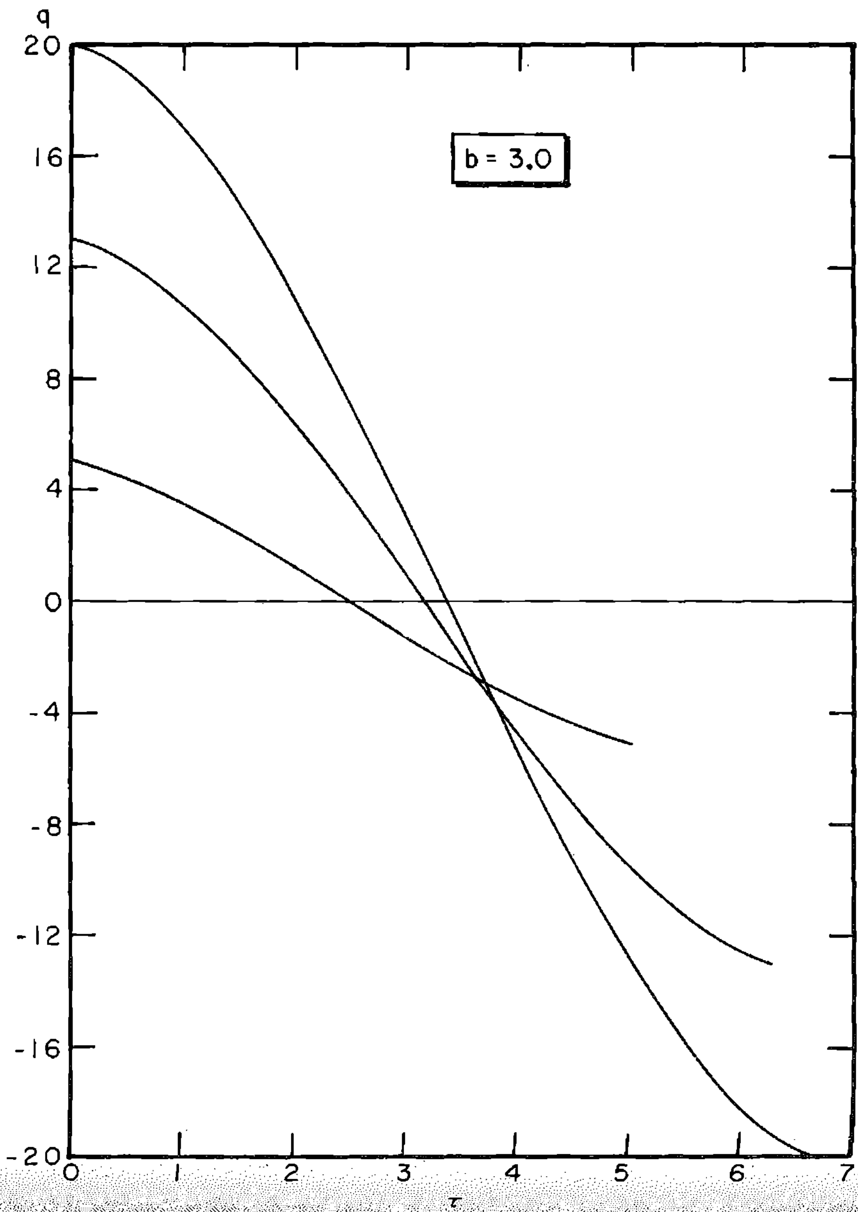

Figure 10

is obtained from $\frac{1}{\eta}$, and

$$I = I_o \frac{d\bar{q}}{dt} = - \frac{I_o}{\eta}$$

For the case $q(0) = 13$, $b = 1.5$ the peak current is then about 300 kiloamps. The shape of the waveform seems to correspond to traces shown in the Geneva paper on Zeta.

PROGRESS IN THE ANALYSIS OF THE ASTRON E-LAYER*

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Abstract

The E-layer of the Astron, even in its uniform portion far from the ends, is complicated by the slowing down of the electrons, by their scattering in angle, by the diamagnetism of the reacting plasma and by possible rotation effects from the angular momentum imparted by the repeated influx of the high energy E-layer electrons. As initial steps in an analytical attack two simpler problems have been solved. The first is that of a cylindrical configuration of relativistic electrons, uniformly distributed in azimuth and all having the same canonical angular momentum and energy. This is in a uniform impressed magnetic field which is modified by the E-layer electrons only. The second covers the generalization to a spread in angular momentum but no spread in energy. In both cases by basing the analysis on a delta-function distribution in constant-of-the-motion space (a suggestion of E.G. Harris), the problem has been reduced to the solution of a second-order differential equation in dimensionless variables. The important parameters are (1) the ratio, G, of the radius of the E-layer injection circle to the radius of gyration of an injected electron in the uniform impressed vacuum magnetic field, (2) the ratio h/h_2 of field strength within the layer and interior to it to the impressed field, (3) the ratio s_1/s_2 or t_1/t_2, of the smallest pericenter radius to the injection radius, and (4) the "number", $(2\ r_e/\gamma)$ N, of electrons per unit axial length of layer, where r_e is the classical electron radius, γ is the ratio of relativistic to rest mass of the electron and N is the actual number of electrons/cm. The momentum range and density distribution (in momentum space) are additional parameters in the second problem, but every useful purpose seemed to be served by using the full momentum range accessible to the electrons (with one or two exploratory exceptions) and by using a uniform density distribution.

The solutions to both problems have common characteristics: When G lies between negative infinity and unity, the injection circle is

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*Work was performed under auspices of the U. S. Atomic Energy Commission. Some of this material also is appearing under the title "Self-Consistent Field of Single-Type Electrons in a Uniform Magnetic Field," Physical Review, May 1, 1959.

This paper may also be identified as Report UCRL-5522, Rev.

a locus of pericenters and the E-layer lies wholly outside it. This case is not of physical interest but is covered mathematically by the range of G from unity to infinity. When G is less than 2, no field reversal is possible up to the full limit of electrons which the layer will hold. In this condition

$$(2r_e/\gamma)N_{max} = 2G - 2$$

and the ratio, h_1/h_2, of interior to impressed field is

$$h_1/h_2\Big|_{N\ (max)} = 2/G - 1$$

When G is greater than 2, so that with few electons present, the electron gyrations do not encircle the axis (but with many they will), the same relations apply. The negative value for h_1/h_2 shows field reversal. For values of G in the range 2 to ~4 the transition, with increasing N, from nonencirclement of the axis to encirclement, is accompanied by a discontinuous decrease in h_1/h_2 followed by further decrease into the negative range or within the negative range as N increases toward N_{max}. For values of G in excess of ~4 the h_1/h_2 - vs -N curves are double-valued in h_1/h_2 alone or in both h_1/h_2 and N.

A third problem is in active process of solution. The mathematical formulation is complete. The generalization here is to electrons which are allowed to slow down from their injection angular momentum and energy according to the energy loss, but not scattering, due to the Coulomb fields of the reacting-plasma particles. The diamagnetism of the plasma is still left out. It has still been possible to formulate this very generally, but iteration will be required to obtain the self-consistent field.

The fourth problem will include (it is hoped) scattering as well as energy loss. The diffusion equation in momentum space has been derived, but the actual attack has not yet been formulated.

D. STABILITY

LONGITUDINAL PLASMA OSCILLATIONS IN AN ELECTRIC FIELD

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Abstract

The properties of longitudinal plasma oscillations in an external electric field are investigated. In a completely linear approximation, it is found that the d-c electric field introduces essentially no new effects. A quasi-linear approximation is also considered, in which couplings between different plasma modes are neglected while the space-averaged distribution functions are assumed to be approximately independent of time. In this case, a Maxwellian distribution function is found to be always unstable against the growth of very long wavelength oscillations.

I. Introduction

In the course of an attempt to understand in more detail the possibility, suggested by Buneman,[1] that long range cooperative effects in the form of growing plasma waves may provide a new mechanism for plasma resistivity, we have studied the dispersion equation for longitudinal plasma waves in presence of an external electric field. While we have not, as yet, succeeded in achieving a quantitative understanding of Buneman's mechanism, the results concerning the effect of an electric field on plasma waves are self-contained and may be of value also in other investigations.

We consider a plasma composed of electrons and ions and assume that

[1] O. Buneman, Phys. Rev. Letters, 1, 8 (1958).

the distribution function (in phase space) for each species obeys a collisionless Boltzmann equation, with electromagnetic fields whose sources are the plasma charge and current density. Since the two-stream instability which Buneman considers involves only longitudinal plasma waves, we neglect the magnetic field due to the plasma current. We also assume that no external magnetic field is present. The problem is then essentially one-dimensional, and we have for the electron distribution function, $f(x, v, t)$,

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}(E_e + E)\frac{\partial f}{\partial v} = 0 \ . \tag{1}$$

The ion distribution function, F, satisfies the same equation with $e/m \rightarrow -e/M$. The external electrical field $E_e(t)$ is a given function of time, while the self-consistent plasma field, E, is determined from Poisson's equation

$$\frac{\partial E}{\partial x} = 4\pi e \int dv\ (F - f) \ . \tag{2}$$

If $E_e = 0$, the linearized form of these equations can readily be solved. The resulting dispersion equation[2] predicts Landau damping[3] if the unperturbed distributions have no relative mean velocity and gives growing waves if the mean velocities differ by more than ε times the electron thermal velocity (for $T_i = T_e$), where ε is a number of order one whose exact value[2] depends upon the form assumed for the unperturbed velocity distributions. It is the aim of the present paper to generalize these field free results and to examine the effect of an external electric field upon the plasma waves.

With the usual separation of f into a space averaged part, f_o, and the fluctuations, f_1, around that, we find that in a strictly linear theory f_o must be time-dependent. Consequently, the equation for f_1 does not have harmonic solutions and there is no dispersion equation in the usual sense. This is discussed in Section II. In Section III we consider briefly the consequences of assuming f_o to be time independent, as might be appropriate in a quasi-linear theory which takes account of the effects of the fluctuations upon f_o but neglects the coupling among the fluctuation modes. In this case a dispersion equation of the usual sort can be derived and leads to growing waves with a Maxwellian f_o even in absence of a relative electron - ion drift. We conclude that either the quasi-linear approximation with time-independent f_o is inherently inconsistent or else that it demands a special form for f_o, different in character from a Gaussian.

II. The Linear Theory

It is convenient to make a Fourier expansion of the x dependance of the distribution functions,

$$f(x, v, t) = n_o f_o(v, t) + \sum_k f_k(v, t)\, e^{ikx}$$

[2] J. D. Jackson, "Plasma Oscillations," Physical Research Laboratory Report, December, 1958.

[3] Landau, J. Phys. USSR, 10, 25 (1946).

with similar expansions for F and for

$$E(x,t) = \sum_k E_k(t)\, e^{ikt} \; .$$

Reality of f, E requires $f_k^* = f_{-k}$, etc. The space averaged density of both ions and electrons is indicated by n_o, and f_o is normalized to 1. (The k spectrum is made discrete by using periodic boundary conditions with a period L so that the allowed k values are multiples of $2\pi/L$.) The equations for the Fourier amplitudes are then

$$\frac{\partial f_o}{\partial t} - \frac{eE_e}{m}\frac{\partial f_o}{\partial v} = \frac{e}{mn_o}\sum E_k^* \frac{\partial f_k}{\partial v} \tag{3}$$

$$\frac{\partial f_k}{\partial t} + ikv\, f_k - \frac{eE_e}{m}\frac{\partial f_k}{\partial v} - \frac{n_o e}{m} E_k \frac{\partial f_o}{\partial v} = \frac{e}{m}\sum_{k'} E_{k'+k}\frac{\partial f_{k'}^*}{\partial v} \tag{4}$$

$$ik\, E_k = 4\pi e \int dv\, (F_k - f_k) \; . \tag{5}$$

In the linearized approximation we drop the right-hand sides of Eqs. 3 and 4. Then Eq. 3 is solved by taking f_o to be an arbitrary function of

$$u = v + (e/m)\int_{t_o}^{t} E_e(t') \; . \tag{6}$$

Introducing u and t as independent variables in place of v and t, we can write Eq. 4 as

$$\frac{\partial f_k}{\partial t} + ik\left[u - \left(\frac{e}{m}\right)\int_o^t E_e(t')\, dt'\right] f_k = \frac{n_o e}{m} E_k \frac{df_o}{du} \; , \tag{7}$$

with a similar equation for F_k.

Since the coefficients are time dependent, the solutions of Eq. 7 are not plane waves and we cannot find a dispersion equation in the usual sense. However, we can solve Eq. 7 by using an integrating factor,

$$f_k(u,t) = e^{-ik[ut-\phi(t)]}\left\{ n_o \frac{e}{m}\int_o^t dt'\, E_k(t')\, e^{ik[ut'-\phi(t')]}\frac{df_o}{du} + f_k(u,0)\right\} \tag{8}$$

where

$$\phi(t) = (e/m)\int_{o}^{t} dt' \,(t - t')\, E_e(t') \;.$$

The electron density is then

$$n_k(t) = \int_{-\infty}^{\infty} du\, f_k(u,t) = e^{ik\phi(t)}\left\{ ik\frac{n_o e}{m}\int_{o}^{t} dt'\, E_k(t')\,(t-t') \right.$$

$$\left. \overline{f}_o\left[k\,(t-t')\right] e^{-ik\phi(t')} + \overline{f}_k\left[\,kt, 0\right]\right\} \qquad (9)$$

where the bar denotes a Fourier transform with respect to u,

$$\overline{f}_o\,(\theta) = \int_{-\infty}^{\infty} du\; e^{-iu\theta}\, f_o(u) \qquad (10)$$

$$\overline{f}_k\,(\theta, 0) = \int_{-\infty}^{\infty} du\; e^{-iu\theta}\, f_k(u, 0) \quad ,$$

and an integration by parts has been used to transfer (d/du) from f_o to $\exp\left[iku\,(t' - t)\right]$. Substituting Eq. 9 and a similar expression for ion density into Poisson's equation, Eq. 5, we obtain finally an integral equation for $E_k(t)$,

$$E_k(t) + \omega_p^2\int_{o}^{t} dt'\, E_k(t')\,(t-t')\left\{\overline{f}_o\left[k\,(t-t')\right] e^{ik(\phi-\phi')}\right.$$

$$\left. + \frac{m}{M}\overline{F}_o\left[k\,(t-t')\right] e^{-ikm(\phi-\phi')/M}\right\}$$

$$= (4\pi ei/k)\left[e^{ik\phi}\,\overline{f}_k(kt,0) - e^{-ikm\phi/M}\,\overline{F}_k(kt,0)\right] \qquad (11)$$

where $\omega_p^2 = 4\pi n_o e^2/m$ is the electron plasma frequency.

In absence of the external electric field, $\phi = 0$ and the integral equation is of the convolution type. A solution is readily obtained by means of Laplace or one-sided Fourier transforms,

$$E_k(\omega) = \frac{R(\omega)}{1 + D(\omega)} \qquad (12)$$

where $R(\omega)$ is the transform of the right-hand side of Eq. 11 and $D(\omega)$ is the transform of

$$\omega_p^2\, t\left[\overline{f}_o(k,t) + (m/M)\,\overline{F}_o(k,t)\right]$$

The necessary and sufficient condition for stability of the oscillations is that the denominator of Eq. 12 have no roots in the upper half ω plane. This problem and the properties of $D(\omega)$ have been carefully discussed by Jackson[2].

The integral equation is also simple if only electron fluctuations are considered. In the limit $m/M \to 0$ we have again a convolution equation, this time for the quantity $E\, e^{-ik\phi}$. Since ϕ is real, the stability properties are identical with those in absence of an external field.

For the case where neither m/M nor E_e vanishes, Eq. 11 is rather formidable. For any given initial conditions, the right-hand side of Eq. 11 is known and one could at least obtain a numerical solution. To determine the stability properties, however, it is necessary to decide whether Eq. 11 has solutions with unbounded E for <u>any</u> initial conditions. This information is readily obtained from the usual dispersion equation but we do not know a general technique for extracting it from the integral equation. Some progress can be made by rewriting Eq. 11 in terms of a formal operator representation, as follows. We solve Eq. 7 by formally inverting the differential operator,

$$f_k = \frac{n_o e}{m}\left[\frac{\partial}{\partial t} + ik\,(u - \dot{\phi})\right]^{-1} E_k \frac{df_o}{du}\,. \tag{13}$$

The density is then

$$n_k = \int du\, f_k = \frac{n_o e}{m\, i k}\, g\left[\frac{i}{k}\frac{\partial}{\partial t} + \dot{\phi}\right] E_k \tag{14}$$

where the function g is defined by

$$g(x) = \int_{-\infty}^{\infty} du\, (u - x - i\varepsilon)^{-1} \frac{df_o}{du}\,, \tag{15}$$

the singularity in the integrand for real x being defined in the manner appropriate to an initial value problem. Substitution of Eq. 14 and the analogous expression for ion density into Poisson's equation gives the operator form of Eq. 11

$$E_k = \left(\frac{\omega_p^2}{k^2}\right)\left[g\left(\frac{i}{k}\frac{\partial}{\partial t} + \dot{\phi}\right) + \left(\frac{m}{M}\right) G\left(\frac{i}{k}\frac{\partial}{\partial t} - \frac{m}{M}\dot{\phi}\right)\right] E_k \tag{16}$$

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2. Jackson, op. cit, p. 2.

where G is defined as in Eq. 15 with the ion distribution, F_o, in place of f_o.

If f_o is Maxwellian,

$$f_o(u) = \frac{e^{-u^2/a^2}}{\pi^{1/2} a} \tag{17}$$

then

$$g(\xi) = a^{-2} Z'(\xi/a)$$

where $Z(x)$ is the "plasma dispersion function" which is always encountered in an analysis of plasma oscillations linearized about a Maxwellian distribution,

$$Z(x) \equiv \pi^{-1/2} \int_{-\infty}^{\infty} d\theta\, (\theta - x - i\varepsilon)^{-1} e^{-\theta^2}$$

$$= 2ie^{-x^2} \int_{-\infty}^{ix} e^{-q^2} dq = i\sqrt{\pi}\, e^{-x^2} - 2x\, Y(x) \quad ,$$

$Y(x)$ being real for real x,

$$Y(x) \equiv e^{-x^2} x^{-1} \int_o^x e^{q^2} dq \quad . \tag{18}$$

(For some useful properties of Y and Z, see Jackson[2].) Even in the low temperature limit $(a \to 0)$ Eq. 16 is complicated, for the asymptotic form of Z' is

$$Z'(x) \to -2i\sqrt{\pi}\, x\, e^{-x^2} + x^{-2} \quad \text{for } x \to \infty \quad , \tag{19}$$

giving

$$g(\xi) \to \xi^{-2} - 2i\sqrt{\pi}\, (\xi/a^3)\, e^{-\xi^2/a^2} \tag{20}$$

Instead, we shall use the simpler function,

$$g(\xi) = (\xi + ia)^{-2} \tag{21}$$

[2] Jackson, op. cit, p. 2.

which corresponds to the choice of a resonance shape distribution function

$$f_o(u) = \pi^{-1} a \left(u^2 + a^2\right)^{-1} . \tag{22}$$

For the case where the two species have equal velocity spreads and equal masses $(m = M,\ a = A)$, the equation for E_k is then

$$E_k = \frac{\omega_p^2}{k^2}\left[\frac{1}{\left(\frac{i}{k}\frac{\partial}{\partial t} + \dot{\phi} + i a\right)^2} + \frac{1}{\left(\frac{i}{k}\frac{\partial}{\partial t} - \dot{\phi} + i a\right)^2}\right] E_k \tag{23}$$

The ia term in the denominators, which represents Landau damping for our particular f_o, can be eliminated by the substitution

$$E_k(t) = e^{-akt}\, y(t) . \tag{24}$$

Then Eq. 23 becomes

$$\frac{k^2}{\omega_p^2} y = (\alpha^{-2} + \beta^{-2})\, y \tag{25}$$

where

$$\alpha = \frac{i}{k}\frac{\partial}{\partial t} + \dot{\phi} \qquad\qquad \beta = \frac{i}{k}\frac{\partial}{\partial t} - \dot{\phi} . \tag{26}$$

Rationalizing the denominators in Eq. 25 and setting

$$y = \beta^2 \eta \tag{27}$$

we have finally a fourth order equation for η,

$$(\alpha^2 + \beta^2)\, \eta = (k^2/\omega_p^2)\, \alpha^2 \beta^2 \eta \tag{28}$$

We now specialize to the case of a constant external field. Since electrostatic instabilities tend to be more serious for the longer wavelengths, we first study Eq. 28 in the limit of very small k. An explicit definition of the "small k" regime can be obtained by imagining that the external field is switched off at time t, leaving the two species with velocities $\pm V = \pm e E_e t/m$. The differential Eq. 28 can then be solved with an exponential, e^{iukt}, where u is a root of

$$2(u^2 + V^2) = (k^2/\omega_p^2)\left(u^2 - V^2\right)^2 .$$

The correction to the $k = 0$ solution, $u^2 = -V^2$, is small provided $kV/\omega_p << 1$. Thus, we consider k as "small" if

$$k << m\omega_p/eE_e t \quad . \tag{29}$$

If we define

$$s = kt \quad \text{and} \quad \gamma = eE_e/km$$

then the only explicit occurrence of k is in the factor k^2 on the right side of Eq. 28. In the limit of small k we then have

$$(\alpha^2 + \beta^2)\,\eta = 2\,(\gamma^2 s^2 - \partial^2/\partial s^2)\,\eta = 0$$

whose general solution is

$$\eta = s^{1/2}\,\overline{Z}_{1/4}\,(i\gamma s^2/2) \tag{30}$$

where $\overline{Z}_{1/4}$ denotes any Bessel function of order $1/4$.

The character of the small k solution is now clear. For some choice of initial conditions, the Bessel function in Eq. 30 will involve at least some of the Hankel function of second kind, so that $\eta(s)$ will grow exponentially

$$\eta(s) \approx e^{\gamma s^2/2} \tag{31}$$

for

$$\gamma s^2/2 >> 1 \quad .$$

It follows from Eqs. 27 and 24 that y will have the same growth character as η, while E_k will grow only when the increasing exponential in Eq. 31 exceeds the Landau damping, i. e.

$$\gamma s^2/2 > as \quad ; \tag{32}$$

These results can most conveniently be summarized in terms of three characteristic times:

$$T = m\omega_p/eE_e k \quad ,$$

the time for the field to produce particle velocities of ω_p/k;

$$t_g = \sqrt{T/\omega_p} = \sqrt{m/eE_e k} \quad ,$$

the time at which the Hankel function begins its exponential growth (corresponding to $\gamma s^2 = 1$);

$$t_d = 2ma/eE_e = 2(ka/\omega_p)T ,$$

the time at which Eq. 32 is satisfied and also the time required for the field to produce a relative drift velocity of order a.

For given k, it follows from Eq. 29 that the solution of Eq. 30 is valid only for $t < T$. Thus, there are three possibilities.

(a) If the values of a and E_e are such that

$$t_g < t_d < T$$

or

$$eE_e/kma^2 < 1 < (k_D/2k)^2 \tag{33}$$

(where $k_D = \omega_p/a$ is the Debye wave number), then the Hankel function growth starts at a time (t_g) when its rate is less than the Landau damping. Later on, $(\text{at } t_d)$ but still before $t = T$, the relative drift velocity exceeds a and E_k begins an exponential growth which continues at least until time T.

(b) If

$$t_d < t_g < T, \quad \text{or} \quad 1 < eE/mka^2 < (k_D/k)^2 \tag{34}$$

then even though the relative drift velocity exceeds a at time t_d, growth of E is postponed until the later time (t_g) when the Hankel function attains its asymptotic character. This result is at first surprising; in the case $E_e = 0$ a drift velocity greater than a leads to growth, so that one would here expect growth at a time of order t_d. However, the energy exchange between particles and wave which constitutes the physical reason for growth of the wave cannot occur in a time less than that required for a particle to traverse one wavelength, and this time is just t_g.* Hence we have the double condition for growth in presence of an electric field: t must be great enough for the external field to produce a relative drift velocity greater than the thermal speed and also to accelerate the particles through a distance of at least one wavelength.

(c) If

$$t_g > T \quad \text{or} \quad t_D > T$$

---

*The time for a particle to go a distance $1/k$ in virtue of its thermal velocity alone is greater than t_g when the inequality Eq. 34 holds.

that is, if

$$k > k_D \quad \text{or} \quad eE/mka^2 > (k_D/k)^2 \tag{35}$$

then we can only conclude that no growth of E_k occurs before a time T. Whether it occurs subsequently can only be determined by dropping the restriction to small k or small t.

In the opposite limit of large k or large t, we expect that an approximate solution should follow from setting the right-hand side of Eq. 28 equal to zero. Noting Eq. 27 we then have

$$a^2 y = 0 \tag{36}$$

whose general solution is $y = (c_1 s + c_2) e^{-i\gamma s^2/2}$ where c_1 and c_2 are constants. Thus, y has no exponential growth and the Landau damping, e^{-as}, prevails. The physical reason for the absence of growth is simply that at times greater than T the electric field has accelerated all particles to velocities greater than the phase velocity of plasma waves, ω_p/k, leaving no particles to be trapped by the waves. We see that the general characteristics are just those to be expected from consideration of the field free case, the only new features being the requirement that growing waves occur only if there is time to accelerate a particle through one wavelength, and that after long times $(t >> T)$ waves of a given k stop growing and decay by Landau damping. It seems reasonable to expect a similar behavior in the case $m \neq M$ and also for other choices of f_o, but we have not explicitly demonstrated this.

III. A Quasi-Linear Approximation

We now adopt a different point of view. Instead of assuming the fluctuations to have an amplitude small enough to permit complete linearization, we suppose that as a consequence of Buneman's mechanism a kind of quasi-equilibrium is established in which f_o and F_o are nearly time independent. This can come about only if the amplitudes of the fluctuations have increased to a point where the right-hand side of Eq. 3 approximately balances the term containing E_e. In fact, we would require f_o and F_o to have such shapes as to lead to little growth of the f_k, while also demanding that the f_k have a velocity dependence which enables the nonlinear term in Eq. 3 to cancel the E_e term. It is far from clear whether the equations have any self-consistent solution of this character. As a first step in studying this, however, we have examined the consequences of assuming that

(a) f_o is independent of time.

(b) The nonlinear terms in Eq. 4 can be neglected,

(random phase approximation). At worst, this can be regarded as an approximation to the problem discussed in the previous section, valid over times short compared to that in which f_o changes appreciably

$$t << m \int (dv\, v f_o)/e E_e \quad .$$

Thus, we study the linear system

$$\frac{\partial f_k}{\partial t} + i k v f_k - \frac{e E_e}{m} \frac{\partial f_k}{\partial v} = \frac{n_o e}{m} E_k \frac{\partial f_o}{\partial v} \tag{37}$$

with a similar equation for F_k and with

$$E_k = 4 \pi e \int dv \, (F_k - f_k) \; . \tag{38}$$

We shall assume that E_e is independent of time. The general solution of Eq. (37) (obtained, for example, by straightforward application of the method of characteristics) is then

$$f_k(v,t) = (n_o \frac{e}{m}) \int_o^t dt' \, E_k(t') \Big\{ \frac{d f_o}{dv} (v + \lambda \tau, t') \, e^{-ik(\lambda \tau^2/2 + v\tau)}$$

$$+ e^{-ik(vt + \lambda t^2/2)} f_k(v + \lambda t, 0) \Big\} \tag{39}$$

where

$$\tau = t - t' \quad , \quad \lambda = e E_e / m \; .$$

The electron density is

$$n_k = \int dv \, f_k(v) = (n_o \frac{e}{m}) \int_o^t dt' \, E_k(t') \, i k \tau \Big\{ \bar{f}_o (k\tau, t') \, e^{ik\lambda\tau^2/2}$$

$$+ \bar{f}_k (kt, 0) \, e^{ik\lambda t^2/2} \Big\} \tag{40}$$

where the bar denotes the Fourier transform, defined as in Eq. 10. Upon substituting this and an analogous expression for ion density into Poisson's equation, Eq. 5, we obtain again an integral equation for $E_k(t)$,

$$E_k(t) + \omega_p^2 \int_o^t dt' \, E_k(t') \, \tau \Big[\bar{f}_o (k\tau, t') \, e^{ik\lambda\tau^2/2}$$

$$+ \bar{F}_o (k\tau, t') \, e^{-ikm\lambda\tau^2/2M} \Big] = X(t) \tag{41}$$

where

$$X(t) = (4\pi e/ik)\left[\overline{F}_k(kt, 0)\, e^{-ik\lambda t^2 m/2M} - \overline{f}_k(kt, 0)\, e^{ik\lambda t^2/2}\right] \tag{42}$$

depends on the initial conditions.

When f_o is independent of time, the integral Eq. 41 is of convolution type, and the solution by Laplace transform is immediate. With

$$E_k(\omega) = \int_0^\infty dt\, E_k(t)\, e^{i\omega t} \tag{43}$$

and a similar definition for $X(\omega)$ we have

$$E_k(\omega) = X(\omega)\left\{1 + \omega_p^2\left[r(\omega) + R(\omega)\right]\right\}^{-1} \tag{44}$$

where r and R are transforms of the kernels of Eq. 41,

$$r(\omega) = \int_0^\infty dt\, \overline{f}_o(kt)\, t\, e^{i(\omega t + k\lambda t^2/2)}$$

$$= (-i/k^2)\frac{d}{du}\int_0^\infty d\theta\, \overline{f}_o(\theta)\, e^{i(u\theta + \lambda\theta^2/2k)} \tag{45}$$

with

$$u = \omega/k .$$

$R(\omega)$ is defined in an analogous fashion. In inverting Eq. 43,

$$E_k(t) = \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}\, X(\omega)}{1 + \omega_p^2\left[R(\omega) + r(\omega)\right]} \tag{46}$$

the integral is to be carried out along a contour which passes above all of the singularities of the integrand. Aside from poles of $X(\omega)$, which depend upon the particular initial conditions chosen, the poles of the integrand will occur at points where the denominator vanishes.

$$D(\omega) = 1 + \omega_p^2\left[r(\omega) + R(\omega)\right] = 0 . \tag{47}$$

If Eq. 47, which is just the dispersion relation for this system, has roots in the upper half plane, then $E_k(t)$ will grow exponentially at large times, i.e., the oscillations will be unstable.

To gain some familiarity with the dispersion Eq. 47, we investigate its properties for the particular case of Maxwellian distributions for f_o and F_o. We choose a frame in which the drift velocities are $\pm V$ and we assume both species to have the same temperature,

$$f_o = \frac{e^{-(v-V)^2/a_1^2}}{\sqrt{\pi}\, a_1}$$

$$F_o = \frac{e^{-(v+V)^2/a_2^2}}{\sqrt{\pi}\, a_2} \qquad a_2^2 = (m/M)\, a_1^2 \, . \tag{48}$$

The Fourier transform of f_o is

$$\overline{f}_o(\theta) = \int dv\, e^{-iv\theta} f_o(v) = e^{-\left[a_1^2\theta^2/4 + iV\theta\right]} , \tag{49}$$

and the function r required for the dispersion equation is

$$\begin{aligned} r(\omega) &= \frac{-i}{k^2}\frac{d}{du}\int_o^\infty d\theta\, \overline{f}_o(\theta)\, e^{i(u\theta+\lambda\theta^2/2k)} \\ &= -\frac{1}{k^2}\frac{d}{du}\frac{1}{a_1\mu_1} Z\left(\frac{u-V}{a_1\mu_1}\right) \end{aligned} \tag{50}$$

where

$$\mu_1 = \sqrt{1 - 2i\lambda/k a_1^2} \quad , \quad u = \omega/k$$

and Z is the "plasma dispersion function" defined in Eq. 18. (The reduction of the integral in Eq. 50 to the Z function requires just some completions of the square in the exponent.) The dispersion Eq. 47 is then

$$1 = (\omega_p^2/k^2 a_1^2)\left[\frac{1}{\mu_1^2} Z'\left(\frac{u-V}{\mu_1 a_1}\right) + \frac{1}{\mu_2^2} Z'\left(\frac{u+V}{\mu_2 a_2}\right)\right] \tag{51}$$

where

$$\mu_2 = \mu_1^* \, .$$

The dimensionless parameter $\sqrt{2\lambda/ka^2}$ is just the ratio of the velocity increment produced by the field in a distance $1/k$ to the thermal velocity. In the limit $\lambda = 0$, Eq. 51 reduces, as it should, to the dispersion relation given by Jackson[2]. For $\lambda \neq 0$ but $\lambda/ka_1^2 << 1$ the properties are qualitatively similar to the zero field case. However, for $\lambda/ka_1^2 >> 1$, the character is quite different. In particular, we find that growing waves occur for arbitrarily small values of the drift velocity V, and, in fact, even in the limit $M/m \to \infty$ where the ions are very heavy and do not participate in the oscillations!

Consider the latter case, i. e., an electron plasma with a background of heavy positive ions to provide charge neutrality. We want to know whether the dispersion equation, which now simplifies to

$$\frac{k^2 a^2}{\omega_p^2} - \frac{1}{\mu^2} Z'\left(\frac{u}{\mu a}\right) = 0 \tag{52}$$

has any roots with $\mathrm{Im}\, u > 0$. The use of a Nyquist diagram, as described by Jackson[2], enables us to answer this without the necessity of evaluating Eq. 52 for complex u. Unfortunately, even if u is real, the argument of Z' is complex because of μ, and the separation of Z into real and imaginary parts is simple only when the argument is real, pure imaginary, or proportional to $\sqrt{i}$. We therefore exploit the fact that in the large field limit, $\lambda/ka^2 >> 1$, μ^2 is nearly pure imaginary. Introducing the velocity

$$\gamma = \sqrt{\lambda/2k} = \sqrt{eE_e/2mk} \tag{53}$$

(we shall assume that both k and E_e are positive) we have

$$\begin{aligned} \mu &= \sqrt{-4i\gamma^2/a^2 + 1} \\ &= 2\gamma/a\sqrt{-i}\,(1 + ia^2/8\gamma^2 + \ldots)\ . \end{aligned} \tag{54}$$

If we neglect the a^2/γ^2 term, then

$$\begin{aligned} \frac{1}{\mu^2} Z'\left(\frac{u}{a\mu}\right) = -\frac{a^2}{2\gamma^2}\Bigg\{ & x\sqrt{\frac{\pi}{2}}\left[\sin x^2\left(\frac{2x}{|x|} S\left[x^2\right] - 1\right)\right. \\ & \left. + \cos x^2\left(\frac{2x}{|x|} C\left[x^2\right] - 1\right)\right] + i\left[1 + x\sqrt{\frac{\pi}{2}}\cos x^2\left(\frac{2x}{|x|} S\left[x^2\right] - 1\right)\right. \\ & \left. - x\sqrt{\frac{\pi}{2}}\sin x^2\left(\frac{2x}{|x|} C\left[x^2\right] - 1\right)\right]\Bigg\} \qquad x = \frac{u}{2\gamma} \end{aligned} \tag{55}$$

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[2] Jackson, op. cit, p. 2.

where C and S are the Fresnel integrals

$$C(x^2) + iS(x^2) \equiv \sqrt{\frac{2}{\pi}} \int_0^x e^{it^2} dt \qquad x > 0 \tag{56}$$

For small or intermediate values of $x = (u/2\gamma)$, the representation Eq. 55 is a good approximation for large γ/a. However, in the asymptotic region $(x << -1)$ it is not correct; the real part of μ^2 causes a damping of the linearly divergent, oscillatory character predicted by Eq. 15. To show this, we use the large argument asymptotic form of Z,

$$Z'(x) = \left[-4i\sqrt{\pi}\, x\, e^{-x^2}\right] + \frac{1}{x^2} + \frac{3 \cdot 1}{2\,x^4} + \frac{5 \cdot 3 \cdot 1}{2^2\,x^6} + \ldots$$

where the term in brackets is to be included if and only if $\operatorname{Im} x < 0$. Including the a^2/γ^2 correction to μ in Eq. 54 we then find

$$\frac{1}{\mu^2} Z'\left(\frac{u}{a\mu}\right) = \frac{a^2}{2\gamma^2}\left\{\frac{u}{\gamma}\,\eta(-u)\sqrt{\pi i}\,\exp\left[-\frac{a^2}{\gamma^2}\,\frac{u^2}{16\gamma^2}\right] e^{-i(u/2\gamma)^2} + \frac{2\gamma^2}{u^2}\left[1 + \frac{3}{2}\left(\frac{a\mu}{u}\right)^2 + \ldots\right]\right\} . \tag{57}$$

We see that if $-u/\gamma$ is large compared to 1 but still small compared to γ/a, then the first term of Eq. 57 dominates. It is just the asymptotic form of Eq. 55. For $-u/\gamma$ large compared to γ/a, however, the first term becomes exponentially small (due to the fact that μ^2 is not pure imaginary) and the second term (which is itself tending towards zero) dominates. The first term in the curly bracket of Eq. 57 is proportional, in magnitude, to

$$\sqrt{\pi}\, 2\, x\, e^{-a^2x^2/4\gamma^2}$$

which has a maximum at

$$x = \sqrt{2}\,\gamma/a \ ,$$

the maximum amplitude being

$$2(\gamma/a)\sqrt{2\pi/e} \ .$$

We can now sketch the form of the real and imaginary parts of $1/\mu^2\, Z'(u/\mu a)$ as functions of u (Fig. 1) and hence the form of the Nyquist plot, i.e., the map of the real u axis in the plane of $\mu^{-2}\, Z'(u/\mu a)$ (Fig. 2). As we go from $u = +\infty$ towards $u = -\infty$ along the real u axis

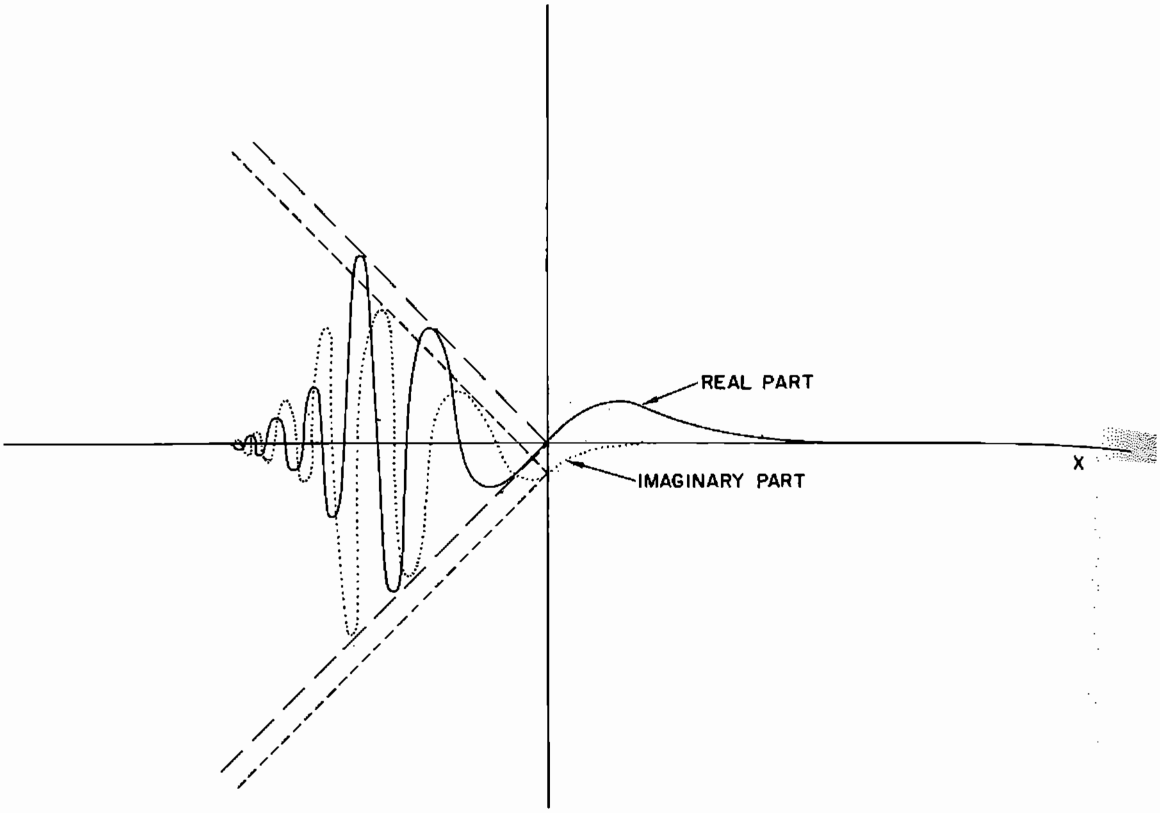

Fig. 1 Real and imaginary parts of $\mu^{-2}Z'(u/\mu a)$ as functions of $x = u/2\gamma$

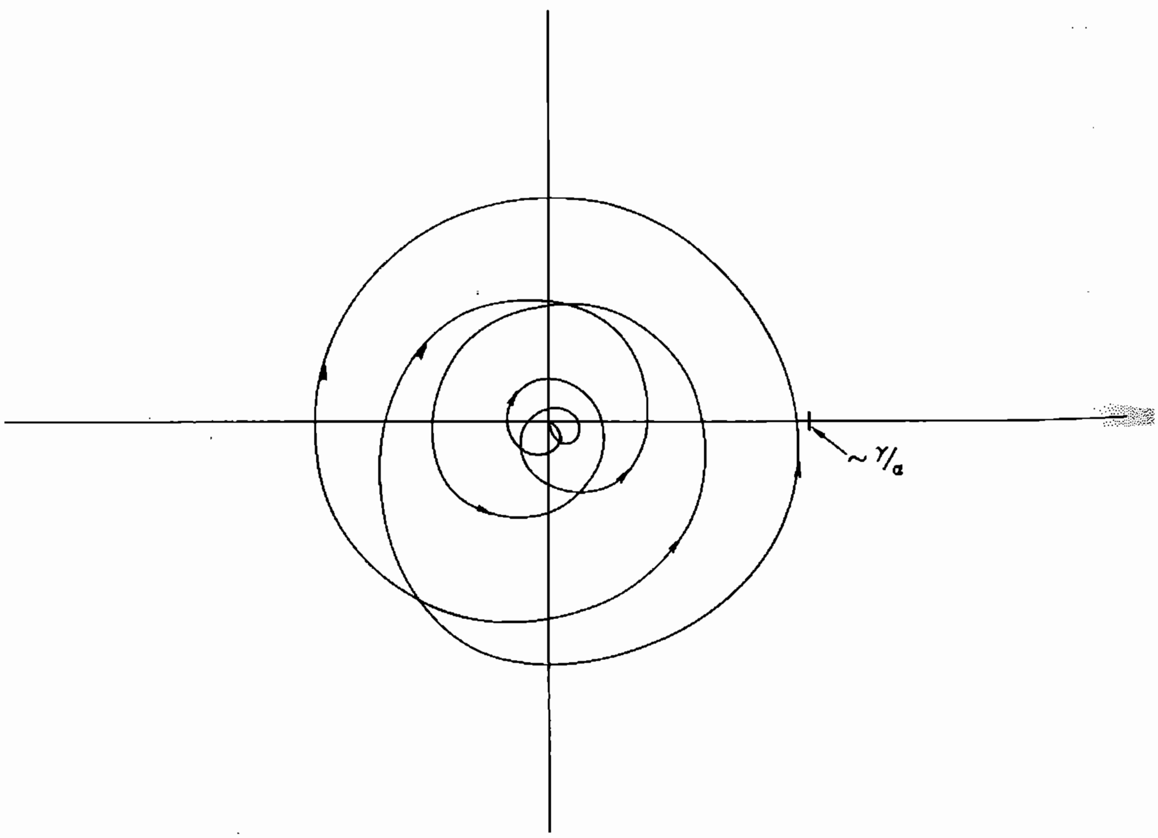

Fig. 2 Nyquist diagram for the dispersion equation (52), corresponding to an electron gas in an external electric field.

(the opposite direction from that indicated by the arrows in Fig. 2). The image point starts from the origin, moves outward in a gradually widening, clockwise spiral until it reaches a radius of order $(a/\gamma)(2\pi/e)^{1/2}$, then quickly spirals back into the origin. The dispersion Eq. 52 will have roots in the upper half plane (leading to growing waves) if this spiral includes (at least once) the point $k^2 a^2/\omega_p^2$. This will happen if

$$\gamma \gtrsim a\,(k_D/k)^2 \qquad k_D \equiv \omega_p/a \; . \tag{58}$$

In order for the large field approximation to be valid, we must simultaneously have $\gamma >> a$. This, combined with Eq. 58, gives as a condition for instability

$$(k/k_D)^2\, n m a^2 << E_e^2 << n m a^2\, (k_D/k)^2 \quad , \tag{59}$$

a condition which can always be satisfied, for nonvanishing E_e, at a sufficiently large wavelength.

This is not a physically reasonable result, since it predicts that an electron plasma with a Maxwellian distribution will have some exponentially growing waves no matter how small the applied electric field. When the ions are assumed to have a finite mass, it is not surprising that the same disease manifests itself and one finds growing waves for an arbitrarily small relative drift velocity. The reason for this difficulty may be that the original hypothesis is inconsistent; there is no solution for which the random phase approximation and the approximation of nearly constant f_o are both valid. At any rate, if a solution of the indicated character does exist, then the present results show that f_o and F_o must have forms very different from a Maxwellian distribution.

As a final point, we recall the remark, made at the beginning of this section, that the present analysis should describe the completely linear problem of Section II, at least during a time in which f_o does not change appreciably. The results found here--instability for any external field--will agree with those of Section II only if we can show that the growth rate is small compared to $e\,E_e/ma$, the rate at which f_o is changing. Such a demonstration can, in fact, be given so that the results of the two sections are not inconsistent.

Conclusions

On the basis of the linear analysis of Section II, we conclude that at least for the special distributions treated there, and probably for more general ones as well, the presence of an external electric field causes no significant changes in the stability character of the linearized plasma waves. If the field is very strong, then it may produce a separation of electron and ion mean velocities greater than the electron thermal speed (thus satisfying the field-free condition for growing waves) before it has carried a particle through one wavelength of the oscillations. In that case, growth is delayed until the particles have gone a distance of order $1/k$ and thus had chance to exchange energy with the plasma waves.

From the results of Section III it appears that if a solution of the complete equations in which f_o and F_o are nearly constant in time exists, it must involve either an f_o and F_o with non-Maxwellian shapes or else must be affected in an important way by the nonlinear terms in Eq. 4 which couple one mode to another.

INSTABILITIES DUE TO ANISOTROPIC VELOCITY DISTRIBUTIONS

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Abstract

If the velocity distributions of the electrons and ions of a plasma are sufficiently anisotropic there exist both longitudinal and transverse unstable waves. These instabilities have been investigated using the Vlasov equations. Most of the work has been done on the longitudinal waves and with the assumption that the coupling between longitudinal and transverse modes could be neglected. Since most of the proposed thermonuclear machines create plasmas with anisotropic velocity distributions these instabilities may have serious consequences.

Rather than write down all the equations involved in these derivations, I think it will be more useful if I explain the equations on which they are based and then briefly discuss a number of instabilities which arise from anisotropic distributions. These calculations are based on the following set of equations. A Boltzmann equation without collision terms for each species of particle in the plasma

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \frac{e}{M}\left[\vec{E} + \frac{1}{c}\,\vec{v} \times \vec{B}\right] \cdot \frac{\partial f}{\partial \vec{v}} = 0 \tag{1}$$

and the equations for the vector and scalar potentials

$$\frac{1}{c^2}\,\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla^2 \vec{A} = \frac{4\pi}{c} \sum e \int \vec{v} f d^3 v \tag{2}$$

$$\frac{1}{c^2}\,\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 4\pi \sum e \int f d^3 v \tag{3}$$

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The summation in the above equations is over the various species of particles (electrons and ions in the usual case). $\vec{E}$ and $\vec{B}$ are determined from $\vec{A}$ and ϕ in the usual way. There is also the Lorentz gage condition

$$\nabla \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0 \tag{4}$$

which is not independent but is a consequence of Eqs. (1), (2), and (3).

The usual assumptions of an infinite homogeneous plasma in a uniform magnetic field is made. The equations are linearized and then Fourier analyzed in both space and time. Equation (1) becomes a differential equation in velocity space which can be solved. When f is substituted into Eqs. (2) and (3) a set of four homogeneous algebraic equations is obtained. The dispersion relation is obtained by setting the determinant of the coefficients of these equations equal to zero. In the general case the elements of the determinant will be a rather horrible mess of integrals over the zeroth order distribution function; I will not try to write them down.

We are particularly interested in zeroth order distribution functions which lead to instabilities of the plasma; that is, cause the dispersion relation to have complex solutions for the frequency. We shall try to catalog some of these.

If the wave vector $\vec{k}$ is parallel to the magnetic field (which is taken to be in the z direction) then the terms which couple ϕ to Ax and Ay vanish and the dispersion relation factors into three factors. Setting each factor equal to zero gives

$$1 = \sum \omega_p^2 \int f_o \frac{d^3v}{(\omega + kv_z)^2} \tag{5}$$

and

$$\omega^2 = k^2c^2 + \sum \omega_p^2 \int f_o \left[\frac{\omega + kv_z}{\omega + kv_z \pm \omega_c} + \frac{k^2 v_\perp^2}{(\omega + kv_z \pm \omega_c)^2} \right] d^3v \tag{6}$$

Equation (5) corresponds to longitudinal plasma oscillations along the lines of $\vec{B}$. In this case the magnetic field plays no part. Instabilities are predicted by Eq. (5) whenever there is relative motion between electrons and ions[1] or when two streams of plasma pass through one another. These instabilities are rather well known.

Equation (6) corresponds to transverse circularly polarized waves. Whether the polarization is right-handed or left-handed determines the sign that precedes ω_c. Bernstein and Dawson[2] have discussed the instabilities predicted by Eq. (6) when a stream of charged particles pass through a cold plasma.

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1. O. Buneman, Phys. Rev. Letters, 1, 8 (1958).
2. I. Bernstein and J. Dawson, Papers Presented at the Controlled Thermonuclear Conference, Washington, D. C., 1958, TID-7558, 360 (1958).

An interesting instability has been predicted by Weibel.[3] Consider a distribution of the form

$$f_o \sim e^{-v_\perp^2/\alpha_\perp^2 - v_z^2/\alpha_z^2} \tag{7}$$

Weibel has found that when α_z is sufficiently smaller than $\alpha_\perp$ Eq. (6) predicts instabilities. These instabilities occur even in the absence of an external magnetic field and probably should be called unstable light waves.

Another interesting instability has been found by Rosenbluth.[4] It involves what I think is properly called a resonance between the frequency of a hydromagnetic wave and the cyclotron frequency. In order to see how it comes about let us consider Eq. (7) with $\alpha_z = 0$ and plot a portion of the dispersion relation given by Eq. (7). It has the appearance shown in Fig. 1.

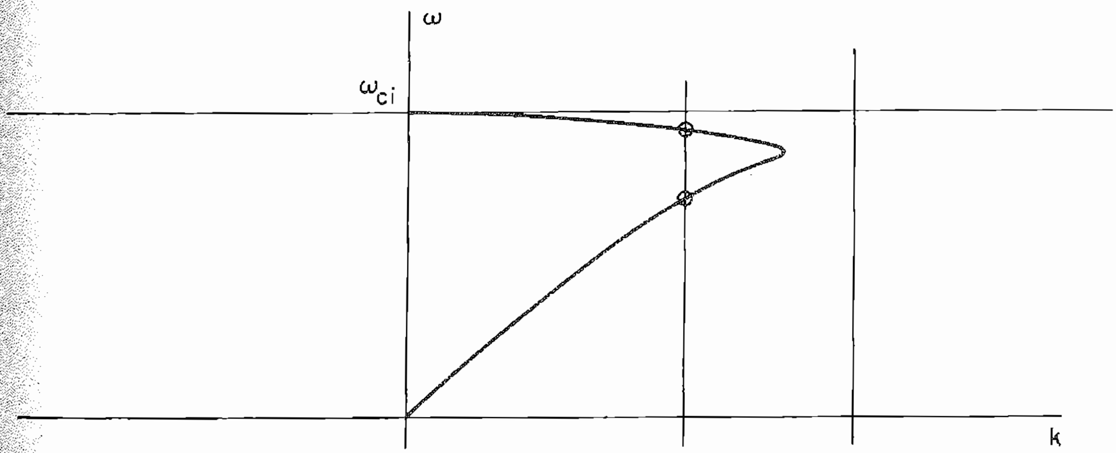

Fig. 1

Near the origin the ω vs. k curve has a slope equal to the Alfven velocity. If $\alpha_\perp$ were zero the curve would approach ω_{ci}, the ion cyclotron frequency asymptotically, but for finite $\alpha_\perp$ it behaves as shown. For small k there will be two real frequencies but as k increases the two frequencies will approach one another until they become equal and for larger values of k the roots become complex. Apparently as the frequency of the wave approaches the cyclotron frequency it becomes possible for the ions to feed their kinetic energy into the wave motion. The wave we have been discussing is the one whose electric vector rotates in the same direction that the ion rotates. The wave with the opposite circular polarization shows a similar behavior in the neighborhood of the electron cyclotron frequency.

Finally we get to the work I have done on the problem.[5,6] If we no longer assume that $\vec{k}$ is parallel to $\vec{B}$ but allow it to have a perpendicular component, then we find that the dispersion relation can no longer be factored. The

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3. E. S. Weibel, Phys. Rev. Letters 2, 83 (1959).
4. M. Rosenbluth, "Recent Theoretical Developments in Plasma Stability," paper presented Nov. 1958 at the San Diego meeting of the Fluid Dynamics Division of the American Physical Society.
5. E. G. Harris, Phys. Rev. Letters 2, 34 (1959).
6. E. G. Harris, Unstable Plasma Oscillations in a Magnetic Field, ORNL-2728 (1959).

terms coupling the longitudinal and transverse waves are found to be of the order of the ratio of the phase velocity of the wave to the velocity of light. I have assumed this ratio to be sufficiently small that the coupling could be neglected. The results which I shall quote are only valid when this is a good approximation. This approximation is equivalent to using only Poisson's equation rather than the complete set of Maxwell's equations and has been made by a number of writers. In this approximation the dispersion relation becomes

$$1 = \sum \frac{\omega_p^2}{k^2} \sum_{n=-\infty}^{+\infty} \int d^3v \left\{ \frac{\omega_c}{v_\perp} \frac{\partial f_o}{\partial v_\perp} \frac{n J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega + k_z v_z + n\omega_c)} + k_z \frac{\partial f_o}{\partial v_z} \frac{J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right)}{(\omega + k_z v_z + n\omega_c)} \right\} \tag{8}$$

where J_n is a Bessel function of order n.

Suppose we have distribution functions of the form of Eq. (7) with $\alpha_z = 0$. Then the integrals in Eq. (8) can be carried out and the dispersion relation written in the form $1 = Y(\omega)$ where $Y(\omega)$ has the appearance shown in Fig. 2.

$Y(\omega)$ has singularities at multiples of the ion and electron cyclotron frequencies. There will be complex roots unless the horizontal line labeled 1 intersects all the loops of $Y(\omega)$ (that is, has two intersections between each

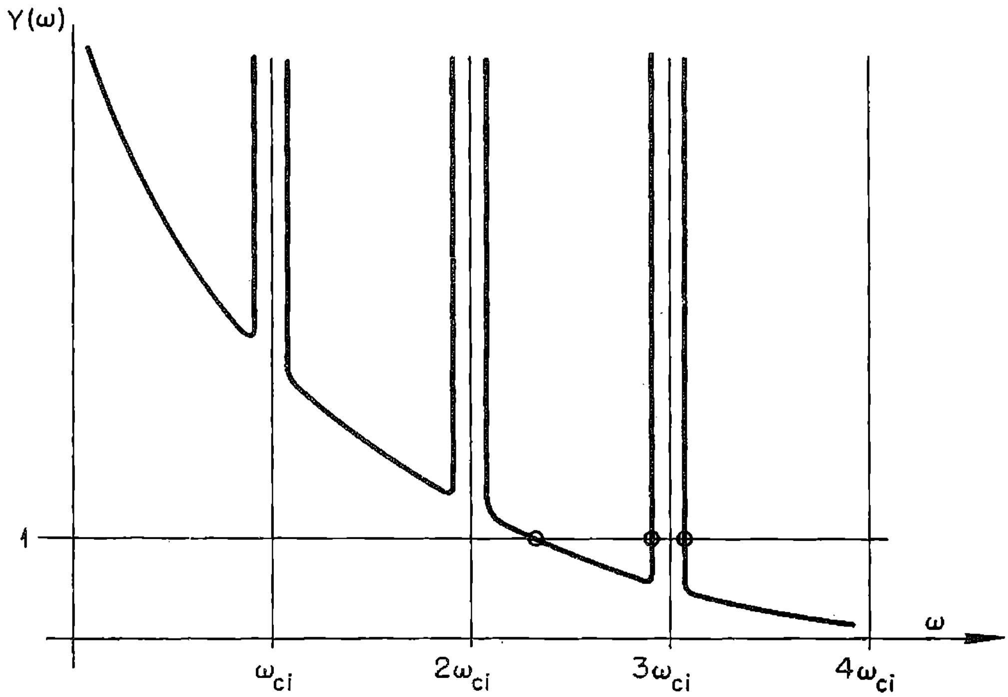

Fig. 2

pair of multiples of ω_{ci} in Fig. 2). The criterion for instability can be written approximately

$$\omega_{pe}^2 > \omega_{ci}^2 \tag{9}$$

or

$$N > \left(\frac{m}{M}\right) \frac{B^2}{4\pi Mc^2} \tag{10}$$

For a magnetic field of about 10^4 gauss this gives $N > 10^7$ particles/cm^3, which is an extremely low density. Of course, in making this calculation I assumed no spread in velocities along the field. Any such spreading will tend to damp out these oscillations.

It would be nice to quote some experimental verification of the existence of these instabilities. There are some experiments by Alfven et al.[7] on trochoidal electron beams which seem to show these instabilities. In these experiments the ions play no part and the instability criterion becomes $\omega_{pe}^2 > \omega_{ce}^2$ or $N > B^2/4\pi mc^2$. Indeed there does seem to be at least order of magnitude agreement with this criterion, but the dimensions of the beam was not measured, so no careful quantitative comparison is possible.

Chairman Northrop: Any discussion?

Dr. Post: On this last point you published a letter in Physical Review a month or so ago in which I understood that you elaborated on the instabilities associated with the last condition.

Dr. Harris: That is true. In that calculation I didn't take into account the ions. So this is the condition.

Dr. Post: Do I understand you correctly then, you would interpret that there are really two cones, one associated with the ions and then much farther out one associated with the electrons?

Dr. Harris: That is true. If you get way out here where the electron cyclotron frequencies appear then you get other instabilities occurring.

Dr. Post: I mean they might even be separated experimentally if they were observable by this criterion you propose.

Dr. Harris: Well, if you go to this one in density then, of course, you are in the unstable region here. You would have to have something suppressing these oscillations.

Dr. Post: That is a weak limit.

Dr. Harris: Yes, every machine would be terribly unstable.

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7. Alfven, Lindberg, Malmfors, Wallmark, and Astrom, Kgl. Tek. Hogskol. Handl. No. 22 (1948).

Dr. Auer: I disagree with Harris. If you look at the sort of thing he wrote down you will find that the velocity of light cancels out in a couple of terms during the transverse and longitudinal and what appears is the ratio of the phase velocity to, let's say, the thermal velocity supplementing the average velocity. Furthermore, if you are going to make some sort of coupling, then I believe the only way you can try to make this deep coupling make sense is if you assume that your frequencies are large compared, say, to the plasma frequency or the cyclotron frequency, and one place where you certainly must make a big coupling is right around the ion cyclotron frequency.

Dr. Harris: Perhaps I didn't look closely enough for the cancellation you spoke of.

Dr. Rosenbluth: Do you have a comment on Auer's question?

Dr. Harris: As I said, I didn't find it but perhaps I didn't look close enough.

Dr. Bernstein: I disagree with Auer, only to look at the maximum equation in the term Fe sub EDT.

Dr. Auer: What I meant is the Boltzmann--

Dr. Bernstein: This has nothing to do with the Boltzmann equation. It has only to do with the Maxwell equation. Under what circumstances can you write the terms?

Dr. Auer: There is another point. The question is when can you disregard the $\vec{v} \times \vec{B}$ term?

Dr. Bernstein: I think that is entirely separate.

Dr. Auer: How so?

Dr. Bernstein: Because the distinction between longitudinal and transverse is contingent only on whether or not the velocity is zero.

Dr. Auer: Yes, but it so happens that the v x B term in the equation brings in an additional coupling and you find that you cannot, for instance, in the region that Harris spoke of, make the decoupling. The $\vec{v} \times \vec{B}$ terms have not been treated properly in the equation for the distribution.

Dr. Harris: They have been treated properly. The dispersion relation I get down here is exact. The only question is whether I have done the right thing in getting these terms, and I think the only thing to do is to multiply out the determinant and see if there is the cancellation you speak of.

Dr. Rosenbluth: I have just one comment on this resonance instability that you spoke of. It is interesting to notice that this instability occurs for any anisotropic distribution, even if it is just a little tiny bit anisotropic. Of course, the rate becomes extremely small. Even if you have two different temperatures in the perpendicular direction, even if you make them arbitrarily close together you still get the instability. It becomes exceedingly small but nonetheless it does exist for any anisotropic conditions. Then I wanted to ask you, is this condition which you wrote down necessary and sufficient?

Dr. Harris: This is a sufficient condition and actually the factor in here is of this order of magnitude.

Dr. Rosenbluth: You assume both distributions of this form?

Dr. Harris: That is true, yes.

Dr. Allis: Combining this condition with the condition that the frequency is less than cyclotron frequency puts the problem in the upper righthand corner of my diagram. In one corner the wave surface is this way. There are directions along here of zero velocity and those are the only directions that E is parallel to $\vec{k}$. The things are properly coupled here and the B is taken into account. The velocity of the wave is going to zero and that is why they are parallel. So I think it is entirely consistent.

Chairman Northrop: Any more questions?

THE BREAKING OF FINITE AMPLITUDE PLASMA OSCILLATIONS

John Dawson
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Abstract

Large amplitude, plane, electrostatic oscillations of a cold plasma were followed numerically. The amplitude was taken to be just slightly larger than that at which the waves begin to break. It was found that the order wave motion was largely converted to individual particle motions during the first few oscillations. About 50% of the wave energy was lost in two oscillations. A few particles were found to be accelerated to very high energies (of the order of ten times the average energy).

If one goes to a Lagrangian coordinate system, it turns out that there are certain finite amplitude longitudinal electron oscillations of a cold plasma which can be analyzed exactly. For amplitudes greater than a certain critical amplitude, the exact analysis breaks down and the waves exhibit a breaking phenomenon roughly analagous to that of ocean waves. In this paper I want to describe some numerical calculations which I have carried out on this breaking process.

Figure 1 illustrates how the analysis goes. Here we are considering plane plasma oscillations. The plasma is taken to be infinite in extent. The ions are assumed to constitute a uniform, fixed neutralizing background. Let the oscillations be in the x direction, and let all particles in a given x plane execute identical motions. Suppose that the equilibrium position of a plane is x_o and let its displacement be $X(x_o)$. In moving the distance $X(x_o)$ the plane passes over an amount of positive charge equal to

$$e\, n_o\, X(x_o)$$

per unit area. Here n_o is the equilibrium number density of electrons. Now, if the ordering of the electrons is maintained then all the electrons which were on the right of the x_o plane remain on its right and all those which were originally on its left remain on its left. In the equilibrium position there is no net charge on either side of a plane so that now there

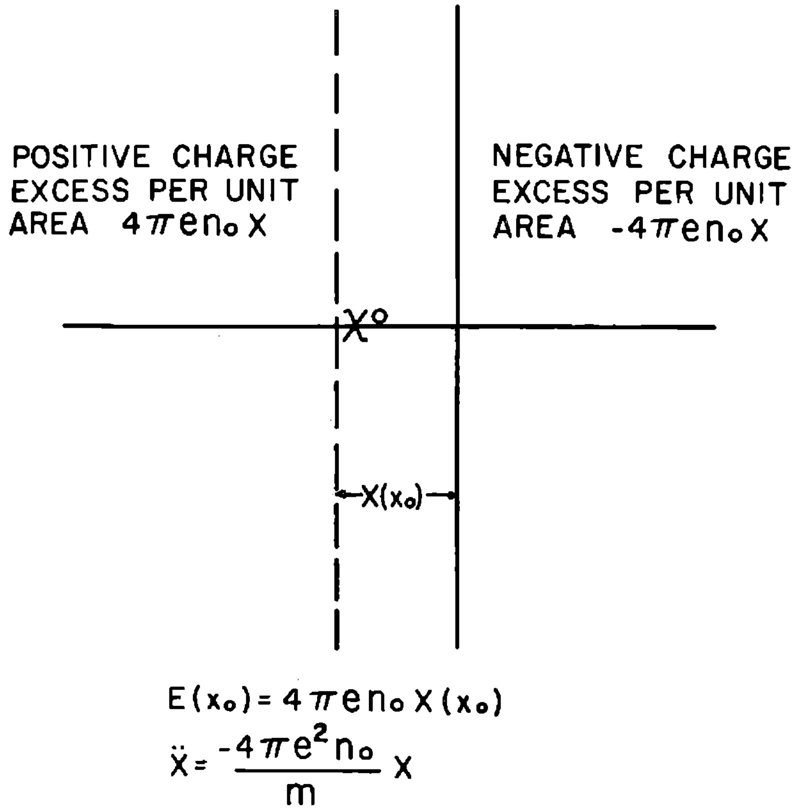

Figure 1

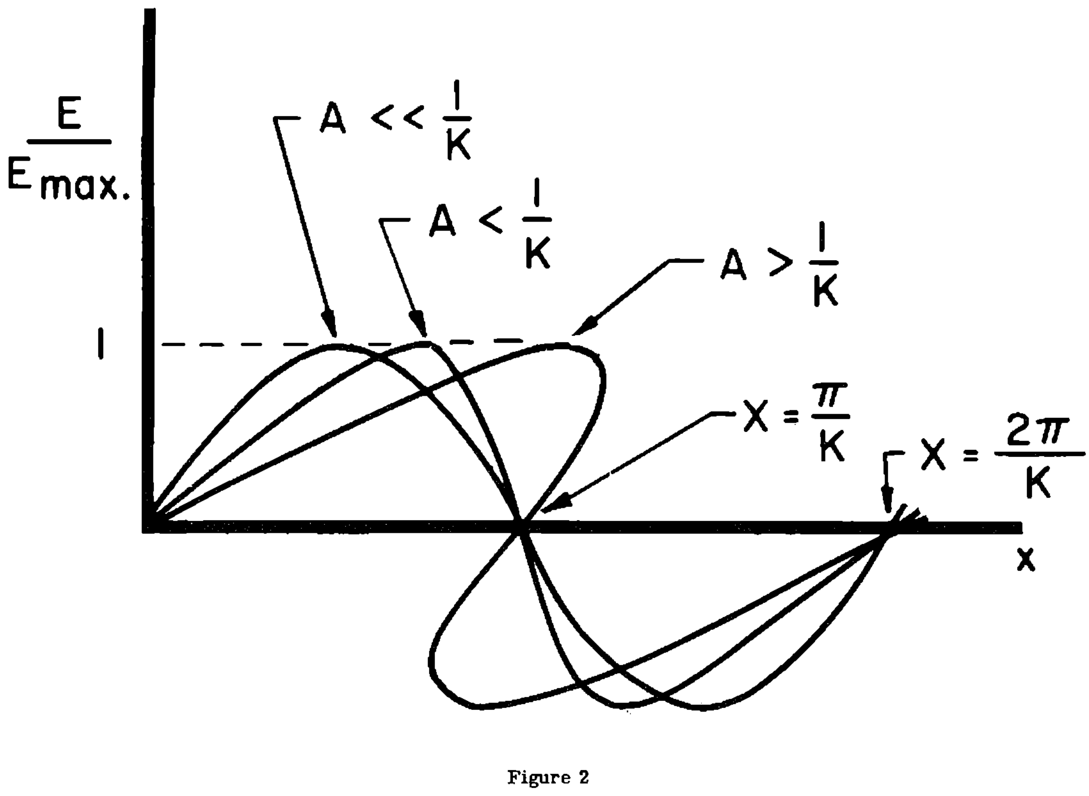

Figure 2

must be a positive charge

$$e\, n_o\, X(x_o)$$

per unit area on one side and a negative charge

$$-\ e\, n_o\, X(x_o)$$

on the other side of the plane considered. We assume that no charges enter or leave the system at plus and minus infinity.

From Gauss' theorem the electric field felt by the plane is

$$E = \frac{4\pi\ e\, n_o\, X(x_o)}{a} \tag{1}$$

and its equation of motion is simply

$$X = \frac{-\,4\pi\ e^2\, n_o}{m}\ X = -\omega_p^{\ 2}\, X \tag{2}$$

This is the equation of motion for a simple harmonic oscillator. Each plane simply oscillates about its equilibrium position independent of what the other electrons are doing provided the ordering of the electrons is maintained.

The most general solution to this equation is given by

$$X(x_o,\ t) = X_1\,(x_o) \sin \omega_p t + X_2\,(x_o) \cos \omega_p t \tag{3}$$

An interesting special example is obtained by setting

$$\begin{aligned} X_2(x_o) &= A \sin K\, x_o \\ X_1(x_o) &= 0 \end{aligned} \tag{4}$$

We may find the electric field as a function of position by making use of equation (1) and the fact that the x_o plane is at the position

$$x = x_o + X(x_o) \tag{5}$$

Plots of E/E_m as a function of x for t = 0, for various values of A are given in figure 2. E_m is the maximum value of E.

For small amplitude waves you get simply a sine wave. As the amplitude gets larger, the maximum and minimum move together. For very large amplitude waves, you find double valued curves. This is, of course, impossible. The derivation has broken down and the ordering of the electrons is not maintained. It appears that this crossing process will create a chaotic situation which will rapidly destroy the wave.

In order to follow the crossing in detail I coded the problem for the Matterhorn 650. The electrons were divided into a discreet number of identical charge sheets. The ions were still taken to constitute a uniform fixed background. The equilibrium situation for this model has the sheets equally distributed in x, that is the sheets are equally spaced. For low amplitude oscillations the sheets oscillate about their equilibrium position with the plasma frequency. For large amplitudes the sheets cross each other and you get the breaking of the oscillations. To find the electric field felt by a sheet, one needs to compute the net charge on each side of it, (number of sheets plus background charge) and make use of Gauss' theorem.

The calculations were carried out for an initial situation in which the velocities of the sheets were a sinusoidal function of their equilibrium positions and for which the amplitude was just slightly larger than that required to give breaking. I should mention that Buneman sent us a preprint of a paper in which he gives the results of a very similar calculation. He followed the breaking of the unstable oscillations produced by a stream of fast electrons passing through a background of heavy, but movable ions.

The calculations were carried out for 10, 30, and 45 particles per half wavelength. Because of the symmetry of the problem, one need only follow the motion of a half wavelength. One must put rigid reflecting walls at the ends since every time a particle leaves the half wavelength region a particle enters with the negative velocity. Further, when two sheets cross each other they may be thought of simply as interchanging equilibrium positions. This fact was made use of in the calculations.

Figure 3 shows the electric field felt by 30 and 45 particles as a function of their instantaneous equilibrium position after just about one half oscillation. The dashed line is the curve for 30 particles and the solid curve is that for 45. Up to particle 25, for the 45 particle case, the curve is smooth and from there on it is ragged. This is due to the particles reflected from the wall which is at position 46 or if one prefers from the particles entering the region from the adjoining half wavelengths. The reflected wave is rather ragged and agreement between 30 and 45 particles is only rough. This indicates that the results might be changed somewhat if one uses more sheets. Nevertheless, quantitative results are probably not too bad.*

Figure 4 shows the electric field for 45 particles as a function of instantaneous equilibrium position for a time of 17.4 radians or about 2.75 oscillations. The dashed curve shows the electric field that would exist if the wave had not broken. As can be seen, the amplitude of the wave has come down by a considerable factor.

Figure 5 shows the velocity of the particles after 17.4 radians. This curve is much more ragged than the one for the electric field and this is due to the fact that the plasma tends to smooth out the density fluctuations of particles, but has no tendency to smooth out the velocity fluctuations. Most of the velocities are fairly low as they should be since this is at a maximum of the electric field. However, a few particles have large velocities and these contain a large portion of the initial wave energy.

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* Further calculations with 90 sheets per half wavelength carried out since this talk was presented bear this out.

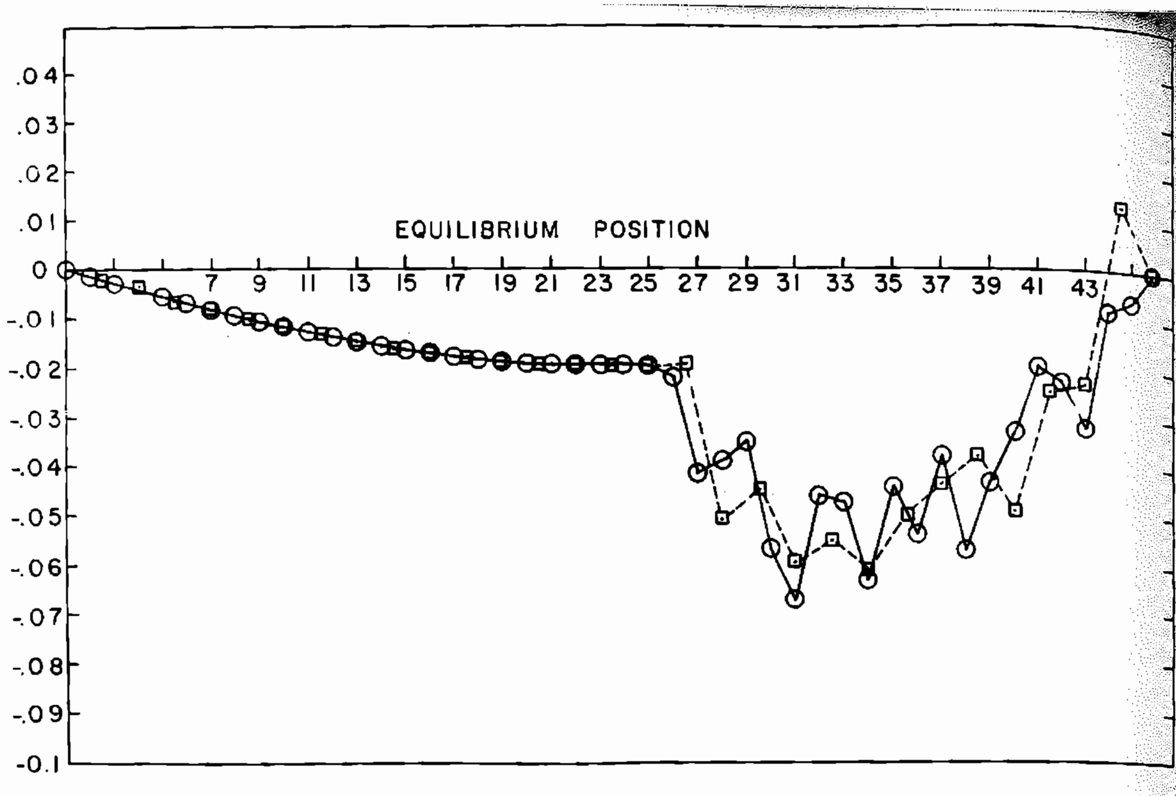


Figure 3

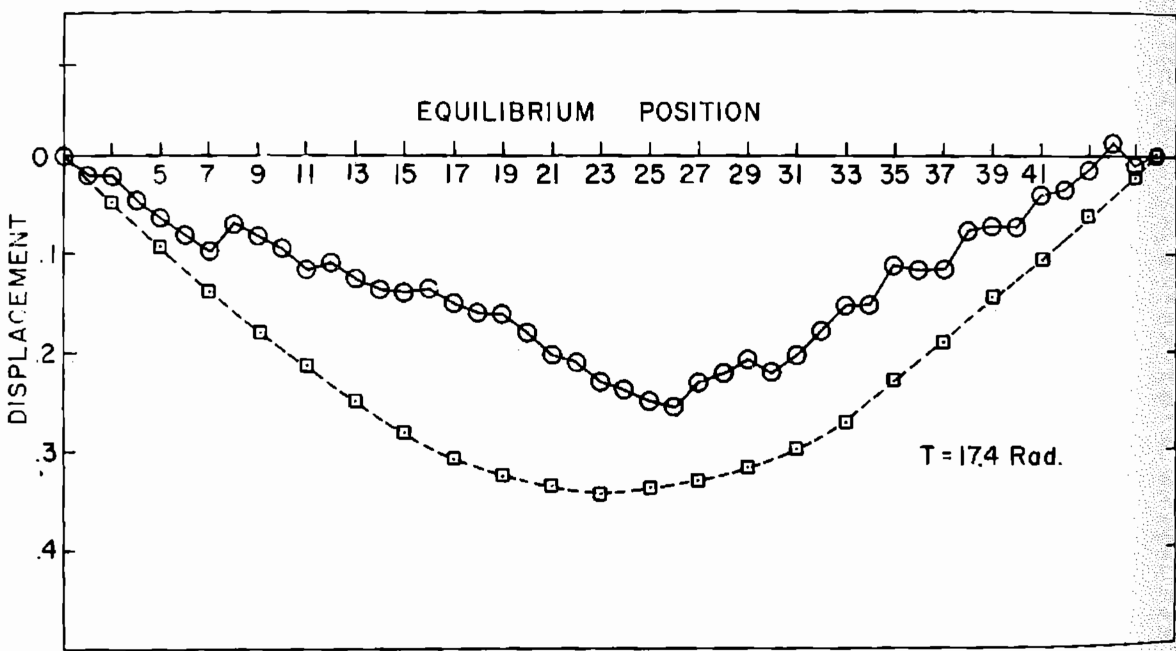
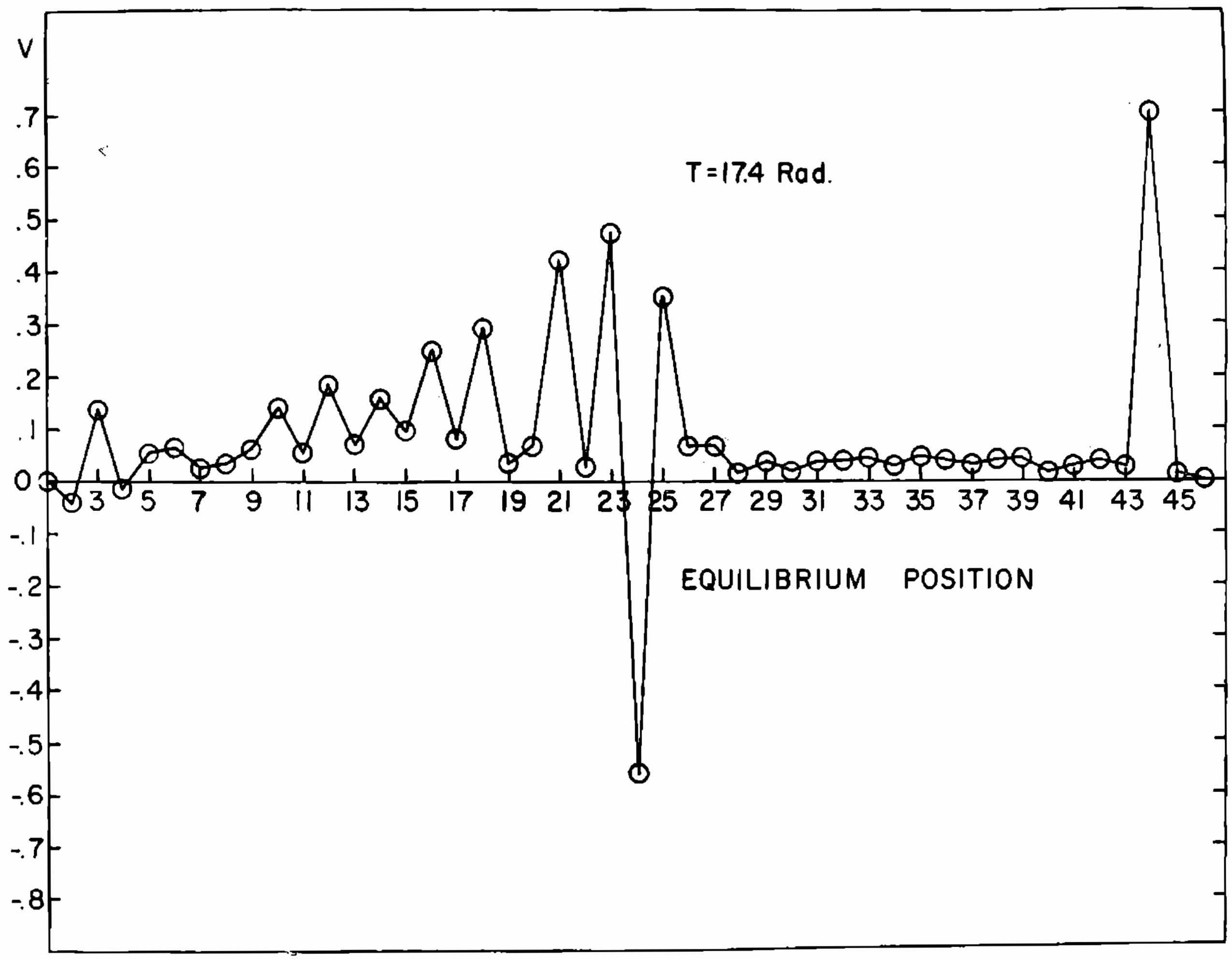


Figure 4


V
.7
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3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45
T=17.4 Rad.
EQUILIBRIUM POSITION

Figure 5

It is clear from Figures 3, 4, and 5 that the ordered wave motion is being randomized. To get an idea of the extent of this, the displacement and velocity were Fourier analyzed in terms of the equilibrium position. Figure 6 shows the absolute amplitudes of the various Fourier modes. The solid curve is the amplitude for the velocity analysis while the broken curve is 10 times the amplitude for the displacement analysis. The hyperbolic curve is the maximum amplitude a mode could have without breaking if the other modes were not excited. The first point on the displacement analysis curve has been left off since it would be at 59. However, one times the displacement analysis curve would be at 5.9 which is not so much larger than the amplitudes of the velocity analysis.

This figure shows that the high harmonics are roughly equally excited and that there is no tendency to feed energy into any particular mode. The velocity amplitudes are roughly 10 times the displacement amplitudes which again shows the tendency toward smooth density curves, but not for smooth velocity curves. This figure also shows that the modes higher than the seventh or eighth will break on every oscillation.

Figure 7 shows the ratio of the energy is the fundamental to its initial value as a function of the time in plasma oscillation periods. The solid curve is for the calculations described above. Initially, it is one. At one plasma period it is about 98 percent; at 1.5 periods it is about 96 percent. It then starts down very fast, going down to about 65 percent after 2 periods. It then raises slightly, then drops again and finally ends up at about 40 percent after 3 periods.

During the first few periods the oscillation is picking up a little random motion from the breaking at the ends. Once the wave has developed a little disorder, the damping seems to go very fast. This indicates that a little random motion or temperature would greatly influence the damping rate. I, therefore, started the wave out with a little random motion. The random energy was about 10 percent of the wave energy. The wave velocity was about 10 times faster than the root mean square of the random velocities and was 3.5 times larger than the maximum random velocity of any particles. Thus, there should have been no Landau Damping in the usual sense. The dashed curve in Figure 7 gives the ratio of energy in the first harmonic to its value at time zero. As can be seen, it damps very fast and in about one plasma period, it is down to only 11 percent of its initial energy. While 45 particles are not very many, this calculation gives a clear indication that thermal motions have a profound effect on the breaking process, greatly enhancing the speed of dissipation.

Chairman Northrop: "Any questions?"

Dr. Dreicer: "Did you notice any periodic nature of the solutions after breaking occurred?"

Dr. Dawson: "No."

Dr. Rosenbluth: "If you had started with a two-stream instability which had had a reasonable sort of temperature spread, you would have a sort of very rapid dissipation before coming to the catastrophic breaking."

Dr. Dawson: "Yes, I think that is probably true. At least these calculations indicate that temperature has a big influence on the damping speed and you should get damping before you get to the breaking amplitude."

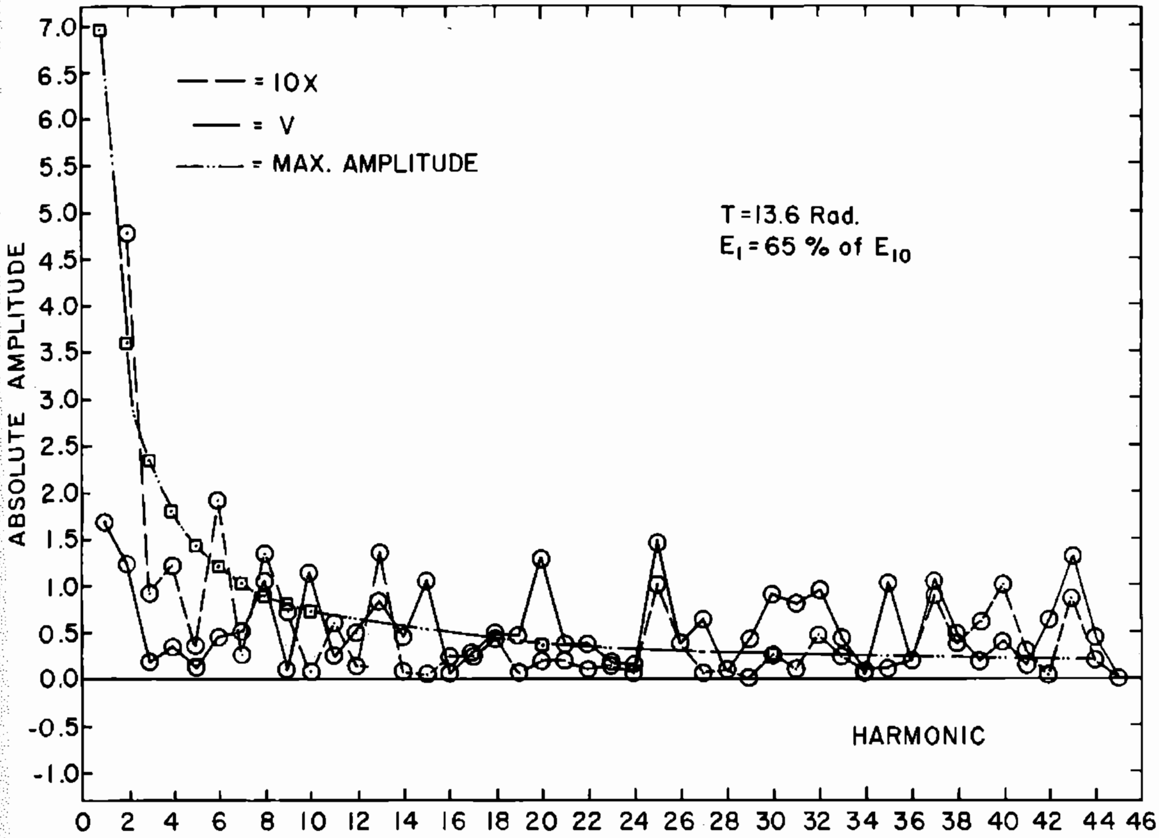

Figure 6

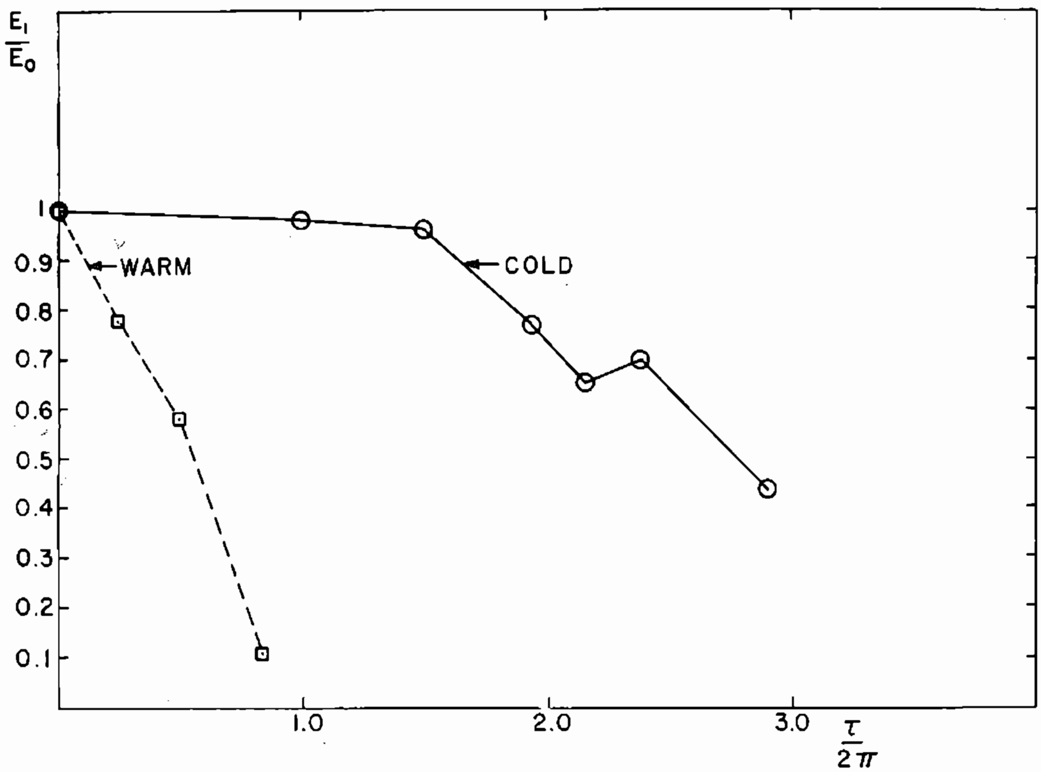

Figure 7

Dr. Tuck: "I take it that you have not just turned the 650 on and let it run just to see what happens."

Dr. Dawson: "No, this calculation is rather slow. With 45 particles, it takes three hours to do one oscillation, so you have to let it run a long time."

Dr. Tuck: "Kruskal will tell you that it will return to the starting conditions."

Dr. Dawson: "I have made some estimates of that time and I find that it will return to the starting conditions after about 10^{45} years with a plasma frequency of 10^{11}. It would take the machine about 16^{60} years to get there which is quite a while.

Dr. Tuck: "In a real gas?"

Dr. Dawson: "No, that was for 45 particles."

Dr. Tuck: "Will 10,000 oscillations with 32 particles?"

Dr. Dawson: "Well, 10,000 oscillations is about 30,000 hours of machine time."

Chairman Northrop: "Were there not some calculations by Ulam several years ago, at Los Alamos on the time it takes to thermalize? The results seem to be that it returned faster than you would have expected, so maybe, it would be shorter than 10 to the 30th years."

Dr. Dawson: "The breaking seems to be more catastrophic than the processes to be considered."

Dr. Longmire: "That is right, there was nothing like breaking in the Ulam problem. There was nothing like losing order; thus you really have lost something."

Dr. Rosenbluth: "Did you see whether the particles were going to anything like a Gaussian distribution?"

Dr. Dawson: "There were always a number of particles at very high energies which sort of ride the wave; so, the distribution would have a much larger tail than a Gaussian."

Dr. Rosenbluth: "They, in themselves, are unstable."

Dr. Dawson: "Yes, they would give up their energy after a while."

EXCITATION OF INSTABILITIES BY RUN-AWAY ELECTRONS

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Los Alamos Scientific Laboratory

Abstract

This paper deals with a fully ionized gas situated in an externally applied electric field E, and investigates its stability to electrostatic disturbances. The linearized Boltzmann equation is solved for the disperion relation between complex ω and real k. Central to this work is the specification of the equilibrium electron velocity distribution. In the weak field regime ($E \ll E_c$) this distribution is obtained by solving the Boltzmann equation in the run-away region of velocity space and joining the result to a solution obtained earlier (H. Dreicer, "The Theory of Run-Away Electrons," 10th Annual Gaseous Electronics Conference, 1957, Cambridge, Massachusetts) for the low velocity body region where collisions dominate. In the run-away region the distribution develops a maximum in the neighborhood of a moving front, ahead of which there are very few particles, and behind which the distribution decays exponentially. This second maximum gives rise to the instabilities we have found.

In the strong field limit ($E \gg E_c$) we have turned our attention to the problem of instability which develops from a non-steady state velocity distribution. The problem is being handled numerically and will be reported if sufficient results are available at the time of the meeting.

STABILITY OF HELICALLY INVARIANT FIELDS ON THE PARTICLE PICTURE

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Abstract

The stability of a system with helically invariant fields has been recalculated using the energy principle based on particle motions developed by Kruskal and Oberman, and also by Rosenbluth and Rostoker. It is found that in the case of isotropic pressure there is no change from the results previously calculated from the hydromagnetic fluid equations. In the case of anisotropic pressure the results are roughly the same unless $p_\perp$ is much greater than $p_{\parallel}$. In this limit the "mirror-type" instability of Newcomb is found.

The stability of helically invariant fields, which are the proposed stabilizing fields for the stellarator, have been calculated [1] on the hydromagnetic fluid picture using the energy principle of Bernstein, Frieman, Kruskal, and Kulsrud[2]. Since this calculation was carried out, two other energy principles have been developed using the particle description in the limit of small particle-gyration radius by Kruskal and Oberman[3], and by Rosenbluth and Rostoker.[4] It seemed advisable to carry out the helically invariant field calculation again on this particle picture, to see if the critical conditions for stability differed from those obtained based on the hydromagnetic picture.

The stabilizing fields can be described basically as follows: Imagine the stellarator stretched out into a cylinder and wires wrapped helically about the cylindrical tube carrying currents in alternate directions.

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1. J. Johnson, C. Oberman, R. Kulsrud, and E. Frieman, Phys. Fluids 1, 281, 1958.

2. I. Bernstein, E. Frieman, M. Kruskal, and R. Kulsrud, Proc. Roy. Soc. 244, 17, 1958.

3. M. Kruskal and C. Oberman, Phys. Fluids 1, 275, 1958.

4. M. Rosenbluth and N. Rostoker, Phys. Fluids 2, 23, 1959.

In addition to the main field, B_o, parallel to the cylinder, a small field, B_1, will be produced which is proportional to $\sin(\ell\theta - kz)$ where k is the wave number of the field in the z-direction and the number of helical wires is 2ℓ. This field produces a rotational transform which depends on radius. It thus has a sheer and is expected to stabilize the interchange instabilities. The magnetic surfaces of these fields are given by

$$r = R[1 + \delta \cos(\ell\theta - kz)] \tag{1}$$

where δ is small and characterizes the amplitude of the superimposed helically invariant fields.

The calculation was made for a very small value of δ and also a very small pressure. The pressure is expressed in terms of the dimensionless number β, the ratio of the plasma pressure to the magnetic pressure. In the hydromagnetic calculation it was found that the system was unstable if β exceeded δ^2.

There were three reasons for repeating the calculation. One reason was the particle picture was more appropriate. The gyration radius is quite small compared to any distance characteristic of the equilibrium. A second reason was that in the original calculation an anomaly appeared at just the radius where one would expect an interchange to occur; namely, the radius at which the rotational transform had a rational value. If β slightly exceeded δ^2 (the critical value) the system was unstable but only to interchanges that were localized to a very small region about this radius of the order of a gyration radius. Either a finite gyration radius theory is necessary to describe this situation, or it is an anomaly due to the hydromagnetic picture, which would disappear in the particle picture. The third reason for repeating the calculation was to treat cases with anisotropic pressure. The hydromagnetic picture assumed isotropic pressure.

For an isotropic pressure distribution the results from the particle picture are identical with those from the hydromagnetic fluid picture. We already knew on the particle picture the system must be more stable.

The anomaly still appears, so that the finite gyration radius is really involved. Thus to do a completely correct stability analysis of these types of hydromagnetic systems, a finite-gyration-radius theory must be developed.

The results for the case of anisotropic pressure depend on $\overline{\beta}$ the ratio of the average of $p_{\parallel}$ and $p_{\perp}$ to the magnetic pressure. This is a little surprising because there is more energy connected with $p_{\perp}$ than with $p_{\parallel}$. Also a new moment θ appears defined as

$$\theta = \frac{m}{4} \int d^3v \, \frac{v_{\perp}^4}{v_{\parallel}} \frac{\partial f}{\partial v_{\parallel}} \tag{2}$$

The results were a slight modification of the results for the isotropic case. In the case $\ell = 3$ the critical $\overline{\beta}$ is

$$\overline{\beta}_c = \frac{\delta^2 \ln\left(1 + \frac{\tau h^2 R^2}{24}\right)}{\frac{\tau h^2 R^2}{24}} \tag{3}$$

where I have assumed θ is proportional to the average pressure and

$$\theta = - \tau \frac{p_{\parallel} + p_{\perp}}{2} \quad . \tag{4}$$

In this case one finds that if $p_{\perp}$ is very large compared to $p_{\parallel}$, θ and τ are very large, and $\overline{\beta}_c$ is very small. This is the situation where the particles have most of their motion perpendicular to the fields. If a perturbation producing a slight weakening in the field is made, the particles are trapped in that region by the "mirror-effect" and tend to enhance the instability. If $p_{\parallel}$ is large compared to $p_{\perp}$ (which is the case if runaways are contributing to the pressure) τ is very small and $\overline{\beta}_c = \delta^2$.

However, $\overline{\beta}$ is related to the average pressure rather than the energy. If one computes the total plasma energy contained by the magnetic field, the isotropic pressure case seems to be the case of maximum energy containment.

A VARIATIONAL PRINCIPLE FOR EQUILIBRIA FROM THE PARTICLE POINT OF VIEW

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Abstract

The equilibrium equations for a plasma from the particle point of view are written down in the small m/e limit according to Chew, Goldberger, and Low. These are discussed in the case of a toroidal geometry with magnetic surfaces. The Boltzman function f depends only on the energy, the magnetic moment, and the magnetic surface. A variational principle equivalent to the full system of self-consistent equations is derived under this constraint in f. It is found to be necessary to introduce one more constraint on the particles (besides the constants of the motion for the particle, the magnetic moment, and the magnetic surface). This is a generalization of the longitudinal invariant.

To determine a hydromagnetic equilibrium properly, it is in general, necessary to solve the Boltzman equation together with Maxwell's equations in a self-consistent manner. However, it is often permissable to neglect collisions entirely and to assume the particle-gyration radius is very small. In this limit the equations of motion for a plasma have been developed by Chew, Goldberger, and Low. [1] This self-consistent system can be shown to be equivalent to a variational principle in toroidal geometries. This variational principle would give a rough idea of what types of equilibrium solutions are possible.

The self-consistent equations for an equilibria with no mass motions are as follows: f the Boltzman distribution function depends only on position r, on the parallel velocity q and on the magnitude of the perpendicular velocity $|v_\perp|$. Expressed in terms of $\underline{r}$ the magnetic moment $\nu = v_\perp{}^2/2\beta$ (where β is the magnitude of the mag-

1. G. F. Chew, M. L. Goldberger, and F. E. Low, Proc. Roy. Soc., 236, 112, (1956).

netic field) and the energy $\nu\beta + q^2/2 + e/m\ \phi$ (where ϕ is the electrostatic potential), f is given by

$$B \cdot \nabla f = 0 \tag{1}$$

There are two such equations, one for the electron Boltzman function and one for the ions. The remaining equations (Maxwell's equations) may be written in the form:

$$n_i = n_e \tag{2}$$

$$\nabla \cdot B = 0 \tag{3}$$

$$(\nabla \times B) \times B = \Sigma\ \nabla \cdot P \tag{4}$$

$$\nabla \times E = 0 \tag{5}$$

$$B \cdot E = \frac{\Sigma\ B \cdot (\nabla \cdot P)/m}{\Sigma\ \frac{ne}{m}} \tag{6}$$

where the summations are over the two types of particles, n is the particle density, and P is the stress tensor. From equation one it follows that the magnetic field must lie on magnetic surfaces which we will assume form a set of nested toroids, and will label by ψ the flux they contain.[2] Therefore, f is a constant on a magnetic surface and we have

$$f = f(\epsilon, \nu, \psi) \tag{7}$$

It should be noted from equation (6) that E is a first order quantity but that it enters in the Boltzman equation (1)---through its electrostatic potential ϕ.

In order to construct the variational principle, it is necessary to define a quantity μ analagous to the longitudinal invariant. For any field B which has magnetic surfaces and for any ν and E we set

$$\mu(\nu, \epsilon, \psi) = \int \beta q \frac{ds}{|\nabla\psi|} = \int \beta \left(\epsilon - \nu\beta - \frac{e}{m}\phi\right)^{\frac{1}{2}} \frac{ds}{|\nabla\psi|} \tag{8}$$

where the integration is performed over the magnetic surface ψ.

The equilibria are just those configurations which make the energy W stationary subject to the following constraints: $f = F(\nu, \mu, \psi)$

2. M. D. Kruskal and R. M. Kulsrud, Phys. Fluids, 1, 265, (1958).

where F is prescribed for the electrons and the ions. B has a set of magnetic surfaces which are nested toroids whose rotational transform is a prescribed function of the flux ψ. $E = \nabla\phi$ where ϕ is any function. F must be prescribed to make the total charge inside any magnetic surface zero. The energy W is the sum of the magnetic energy and the kinetic energy of the particles.

From the existence of this variational principle, it seems reasonable to assume that one may find toroidal equilibria for any prescription of the Boltzman functions of the type $f = F(\nu, \mu, \psi)$ and of the rotational transform as a function of ψ. In some sense this prescription would also characterize the possible types of equilibria.

The constraints employed in the variational principle are just those which would be conserved if the magnetic surfaces changed extremely slowly in time. This is in accordance with the thought experiment of Kruskal and Kulsrud.[2] The analogue of the longitudinal invariant, μ, given by equation (8), is also expected to be conserved in time. Its integrand is only defined at those points where $q < 0$ which is taken as the region of integration. If $q > 0$ everywhere (particles untrapped) or if $q < 0$ somewhere (particles trapped) one can show μ is an adiabatic invariant. In the situation where (in changing B) one passes from the case of a trapped to an untrapped particle, it seems likely that μ is conserved.

2. M. D. Kruskal and R. M. Kulsrud, Phys Fluids, 1, 265, (1958).

ON THE STABILITY OF A HOMOGENEOUS PLASMA WITH NON-ISOTROPIC PRESSURE

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Abstract

The stability of a homogeneous plasma in a homogeneous magnetic field with non-isotropic pressure is investigated by applying the macroscopic plasma equations. It is found that the plasma is unstable if the pressure along the magnetic lines of force is too large compared to the pressure perpendicular to the magnetic field; it is also unstable if the perpendicular pressure is too large compared to the parallel pressure.

In the following the propagation of waves in a homogeneous plasma with non-isotropic pressure is investigated. This problem is also of interest for the stability of such a plasma. The stability of a plasma with non-isotropic pressure has been studied by Rosenbluth[1] by applying the particle picture and by Chandrasekhar, Kaufmann and Watson[2] for a pinch effect configuration starting from the Boltzmann equation.

The macroscopic equations as given by Chew, Goldberger and Low[3] and by Schlüter[4] will be applied, neglecting the heat-flow terms:

$$\rho \frac{d\vec{v}}{dt} = - \operatorname{div} \mathbb{P} + \frac{1}{c} \vec{j} \times \vec{B} \tag{1}$$

1. M. Rosenbluth, unpublished.
2. S. Chandrasekhar, A.N. Kaufmann and K.M. Watson, Proc. Roy. Soc. Lond. A 245, 435 (1958).
3. O.F. Chew, M.L. Goldberger and F.E. Low, Proc. Roy. Soc. Lond. A 236, 112 (1956).
4. K. Hain, R. Lüst and A. Schlüter, Z.f. Naturf. 12a, 833 (1957).

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$$\frac{\partial \rho}{\partial t} = -\operatorname{div} (\rho \vec{v}) \tag{2}$$

$$\frac{dp_\perp}{dt} = -2p_\perp \operatorname{div} \vec{v} + \frac{1}{B^2} p_\perp \vec{B}(\vec{B}\operatorname{grad}) \vec{v} \tag{3a}$$

$$\frac{dp_\parallel}{dt} = -p_\parallel \operatorname{div} \vec{v} - \frac{2}{B^2} p_\parallel \vec{B} (\vec{B}\operatorname{grad}) \vec{v} \tag{3b}$$

$$\frac{\partial \vec{B}}{\partial t} = \operatorname{curl} (\vec{v} \times \vec{B}) \tag{4}$$

$$\frac{4\pi}{c} \vec{j} = \operatorname{curl} \vec{B} \tag{5}$$

$$\operatorname{div} \vec{B} = 0 \quad . \tag{6}$$

Here $\vec{v}$ is the macroscopic velocity, ρ the density, $\vec{B}$ the magnetic field, $\vec{j}$ the electric current density, $p_\perp$ and $p_\parallel$ the pressure components perpendicular and parallel to the magnetic field respectively and $\mathbb{P}$ the pressure tensor with the elements p_{ik} given by

$$p_{ik} = p_\perp (\delta_{ik} - \frac{1}{B^2} B_i B_k) + p_\parallel \frac{1}{B^2} B_i B_k \quad . \tag{7}$$

Assuming a homogeneous plasma and a homogeneous magnetic field and linearizing the above equations, one can finally derive the following equation for the amplitude $\vec{v}_1$ of the velocity:

$$[\omega^2 \rho - \frac{1}{B^2} (\vec{B}\vec{k}) (p_\parallel - p_\perp) - \frac{1}{4\pi} (\vec{B}\vec{k})^2] \vec{v}_1 \tag{8}$$

$$- \vec{k} [2p_\perp (\vec{v}_1 \vec{k}) - \frac{1}{B^2} p_\perp (\vec{B}\vec{v}_1)(\vec{B}\vec{k}) + \frac{1}{4\pi} B^2 (\vec{v}_1 \vec{k}) - \frac{1}{4\pi} (B\vec{k})(\vec{B}\vec{v}_1)]$$

$$+ \vec{B} [\frac{1}{B^2} p_\perp (\vec{B}\vec{k}) + \frac{1}{B^4} (p_\perp - 4p_\parallel)(\vec{B}\vec{k})(\vec{B}\vec{v}_1) + \frac{1}{4\pi} (\vec{B}\vec{k})(\vec{B}\vec{v}_1)] = 0.$$

Here ω is the frequency of the wave with the wave vector $\vec{k}$. From this equation one gets the propagation velocity $V = \frac{\omega}{k}$ as a function of direction for the different kinds of waves.

a) Propagation parallel to the magnetic field.

In this case there exists longitudinal and transverse modes which are not coupled. The propagation of the longitudinal mode is given by

$$V^2 = 3 \frac{p_\parallel}{\rho} \tag{9}$$

This is the propagation velocity of a sound wave with one degree of freedom ($\gamma = 3$; γ = ratio of specific heats). The velocity of the transverse mode is given by

$$V^2 = \frac{p_\perp - p_\parallel}{\rho} + \frac{1}{4\pi} \frac{B^2}{\rho} \tag{10}$$

In this case the Alfvén-velocity is increased or decreased if the pressure component parallel to the magnetic field is smaller or larger, respectively, than the perpendicular component. If the parallel component is too large, the propagation velocity becomes imaginary. This is usually regarded as meaning that the plasma will be unstable. The same condition for instability has been derived in other ways by Rosenbluth[4], Parker[5], as well as by Chandrasekhar, Kaufmann and Watson[2].

b) Propagation perpendicular to the magnetic field.

In this case only a longitudinal wave with the velocity

$$V^2 = 2\frac{p_\perp}{\rho} + \frac{1}{4\pi}\frac{B^2}{\rho} \tag{11}$$

can propagate. This corresponds to a gas with two degrees of freedom ($\gamma = 2$). For this direction the velocity is always real.

c) Propagation oblique to the magnetic field.

In this general case there exist three different modes. One is a pure transverse mode (v_1 is perpendicular to $\vec{B}$ and $\vec{k}$) with the velocity

$$V^2 = \left[\frac{p_\perp - p_\parallel}{\rho} + \frac{1}{4\pi}\frac{B^2}{\rho}\right]\cos^2\vartheta \tag{12}$$

where ϑ is the angle between the direction of propagation and the magnetic field. The two other modes (v_1 is in the plane of $\vec{B}$ and $\vec{k}$) are neither pure transverse nor pure longitudinal. V^2 is given by a quadratic equation:

$$\begin{aligned} V^4 &+ \left[\frac{p_\perp}{\rho}(\cos^2\vartheta - 2) - 2\frac{p_\parallel}{\rho}\cos^2\vartheta - \frac{1}{4\pi}\frac{B^2}{\rho}\right]V^2 + \frac{3}{4\pi}\frac{p_\parallel B^2}{\rho^2} \\ &- \left(\frac{p_\perp}{\rho}\right)^2(1-\cos^2\vartheta)\cos^2\vartheta + 3\frac{p_\parallel p_\perp}{\rho^2}(2-\cos^2\vartheta)\cos^2\vartheta \\ &- 3\left(\frac{p_\parallel}{\rho}\right)^2\cos^2\vartheta = 0. \end{aligned} \tag{13}$$

One can show that all roots of this equation are real; therefore overstability can not occur. Furthermore, there exists at least one root with $V^2 \geq 0$. But if the pressure component perpendicular to the magnetic field is too large compared to the parallel pressure, one root will be negative. Again this is usually regarded as meaning that the plasma will be unstable. In this case perturbations propagating nearly perpendicular to the magnetic field cause the instabilities.

Introducing the dimensionless quantities

$$\alpha = 4\pi\frac{p_\parallel}{B^2} \quad \text{and} \quad \beta = 4\pi\frac{p_\perp}{B^2} \tag{14a, b}$$

---

5. E.N. Parker, Phys. Rev. 109, 1874 (1958).

the limits for stability are given by

$$\frac{\beta^2}{3(1+2\beta)} \le \alpha \le \beta + 1 \tag{15}$$

The inequality on the right-hand side is the condition that the parallel component of the pressure not be too large. The left-hand side gives the condition that the perpendicular component of the pressure not be too large compared to the parallel component. This condition is a weaker condition than the one derived by Rosenbluth.

Dr. Blank and Dr. Grad pointed out later to me that for the above condition of instability the character of the differential equations changes from hyperbolic to elliptic. In this case the initial value problem is no longer well posed. The interpretation as instabilities is not obvious and seems to be difficult. This important question has to be investigated further. I would like to thank Dr. Blank and Dr. Grad for several discussions about this problem.

PRESSURE BALANCE AND STABILITY CRITERIA IN THE MIRROR MACHINE*

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ABSTRACT

For some special cases where the plasma energy density is relatively small compared to the magnetic energy density, solutions to the tensor magnetostatic pressure-balance equation have been constructed for the mirror machine. These solutions can be made approximately to conform to previously derived diffusion equilibrium solutions. The solutions thus obtained can be subjected to various existing stability criteria, in order to derive critical relative plasma pressure values. These critical β values are generally of order 0.25, and thus are probably high enough to lie outside of the range of validity of the low β solutions for which they were calculated.

Some time ago I worked out a simple special solution to the pressure balance equations in the mirror machine just mostly to satisfy myself that such things existed and are not a figment of the imagination. I did not carry it beyond a simple special solution, since I found one that approximately satisfied the diffusion boundary conditions. It is now of some interest to revive this solution and compare it with the requirements of some recent stability criteria.

The equation we are to solve for the magnetostatic pressure equilibrium is

$$\nabla \cdot \mathbb{P} = \frac{1}{c} \vec{J} \times \vec{B}$$

$$\mathbb{P} = \begin{bmatrix} P_{\parallel} & 0 & 0 \\ 0 & P_{\perp} & 0 \\ 0 & 0 & P_{\perp} \end{bmatrix} \tag{1}$$

*This paper may also be identified as Report UCRL-5524.

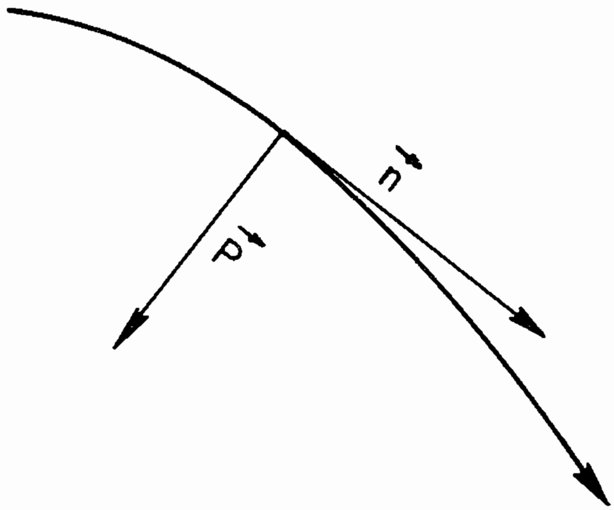


Fig. 1. Unit Vector Diagram.

Choose a set of coordinates with the unit vector $\vec{n}$ along the magnetic lines and the unit vector $\vec{p}$ perpendicular to the lines as shown in Fig. 1. We write down the equations for pressure balance parallel to the magnetic lines and pressure balance perpendicular to the magnetic lines just by solving Eq. 1. Of course, in the parallel direction this side is zero. For pressure balance in $\vec{n}$ direction:

$$\left| \nabla \cdot \mathbb{P} \right|_n = 0 , \qquad \vec{n} = \frac{\vec{B}}{|B|}$$

so that

$$\nabla_n P_{\parallel} = -\left(P_{\perp} - P_{\parallel}\right)\left[\frac{\nabla B}{B}\right]_n \tag{2}$$

For pressure balance $\perp$ to B:

$$\left| \nabla \cdot \mathbb{P} \right|_p = \frac{1}{c} \, J \, B$$

so that, inserting

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{J}$$

$$\nabla_p\left(P_{\perp} + \frac{B^2}{8\pi}\right) = \left[\frac{B^2}{4\pi} + \left(P_{\perp} - P_{\parallel}\right)\right]\frac{1}{R_c}$$

where R_c is the local radius of curvature of field lines, + where concave toward axis, and − convex toward axis.

There are obviously an infinite number of solutions which can be chosen which satisfy the various requirements you wish to put on them. In the low β case one simply ignores the variation of B due to $P_{\perp}$, and is concerned mostly with the solution of Eq. 2. Also I will consider solutions for long machines where curvature terms will not be large.

The coordinate system for the proposed solution is shown in Fig. 2 indicating the field intensity roughly as shown. Let u be the coordinate in along the field lines, running from minus π to plus π. Let B_0 be the value of B at u = 0 and B_M the mirror field. Since we are looking for a low β solution, I will specify the form of B, and choose a particular functional form which roughly fits our experimental cases.

$$
\begin{aligned}
B(u) &= B_0\, e^{-\alpha(\cos u - 1)} \\
&\approx B_0\,(1 - \alpha \cos u)\ , \quad \alpha << 1
\end{aligned}
\tag{3}
$$

This form of field variation approximates actual fields of interest.

We have to make yet another specific choice, that for $P_\perp$, because there are an infinite number of possible solutions. Let us choose one which at least roughly satisfies the mirror loss diffusion equation. Certainly at the peak of the mirrors not only must the pressure go to zero but the first derivative of the pressure must go to zero also. I have therefore chosen a solution which satisfies roughly the slope requirements of the diffusion

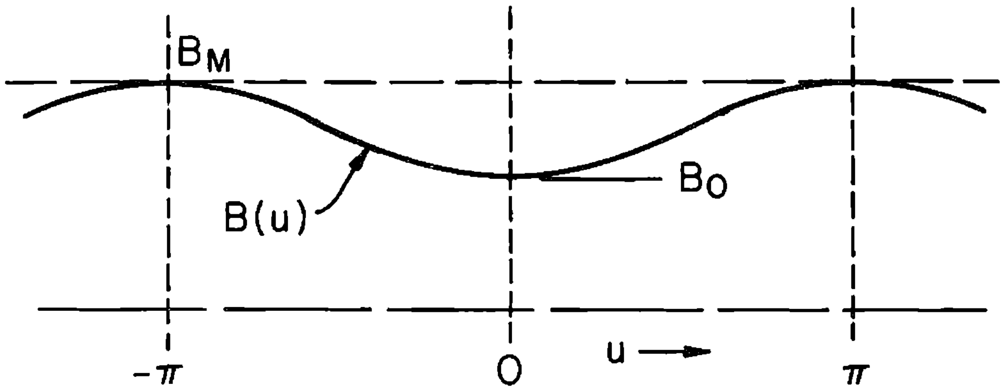


Fig. 2. Coordinate System.

equations such as those calculated by Rosenbluth, Judd and McDonald. The one chosen is given by Eq. 4

Take
$$P_\perp = \frac{P_0}{2}\,(1 + \cos u) \tag{4}$$

From this assumption, of course, finding $P_\parallel$ becomes a completely solvable problem. I shall make a change of variables to simplify.

Let $\quad w = (1 + \cos u)$

Then
$$P_\perp = \frac{P_0}{2}\, w \tag{5}$$

$$P_\parallel = \frac{P_0}{2}\left\{ w - \frac{1}{\alpha}\left[1 - e^{-\alpha w}\right]\right\} \tag{6}$$

Note that the ratio of $P_{\parallel}$ to $P_{\perp}$ at any point is equal to

$$\frac{P_{\parallel}}{P_{\perp}} = \left\{1 - \frac{1}{\alpha}\left[\frac{1 - e^{-\alpha w}}{w}\right]\right\} \tag{7}$$

For large alpha this approaches 1, i.e., a scalar pressure. Note that it really takes an enormous mirror ratio to get very close to 1 because the expression is exponential in the mirror ratio $(R = e^{2\alpha})$.

Taking a particular case, for example $\alpha = 1$, and plotting the solutions, we find that $P_{\perp}$, starting at $u = -\pi$ starts with zero slope. $P_{\parallel}$ starts with a higher order contact. $P_{\perp}$, $P_{\parallel}$, $P_{\parallel}/P_{\perp}$ and B are shown in Fig. 3 and evaluated in Table 1.

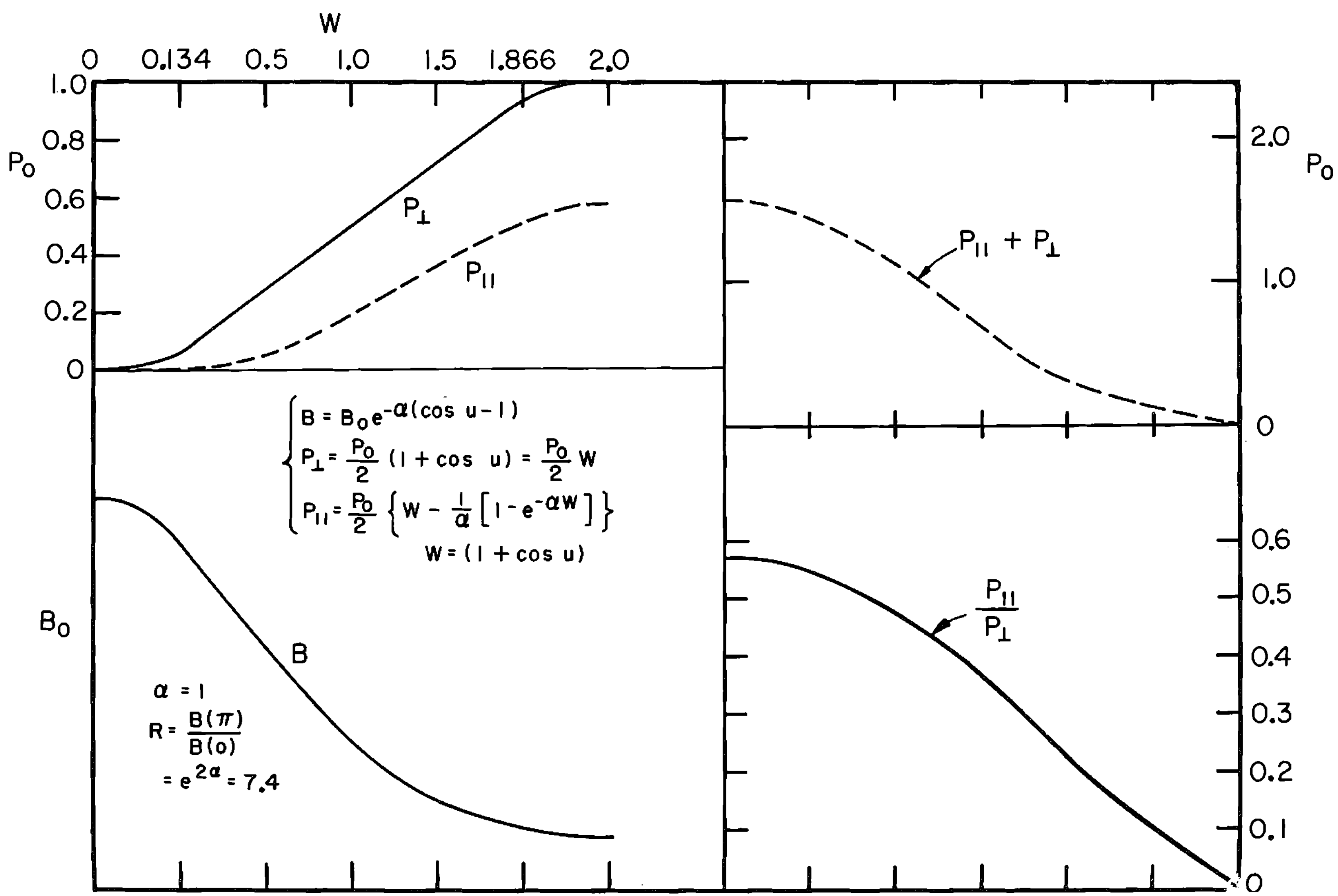
These solutions have the property that they fit the boundary conditions, at least approximately, in a way that is consistent with the solution of the diffusion equation, so that they may have some physical reality.

This solution may be compared with instability calculations for the recent nonisotropic instabilities. Remember that we are dealing not only with a low β case but also with a very long machine. Thus in some sense, if the nonisotropic instabilities are local we need only have a region large with respect to orbit diameters to satisfy the required physical conditions

Table 1. Tabulated Pressure Balance Solutions

u	cos u	(1+cos u)=w	$e^{-\cos u}$	$e^{-(\cos u+1)}$	(w-1)	$2P_{\parallel}$
0°	1	2	0.37	0.136	1	1.136
30°	0.866	1.866	.42	.154	0.866	1.020
60°	0.5	1.5	.61	.224	0.5	0.724
90°	0.0	1.0	1	.368	0.0	.368
120°	-0.5	0.5	1.65	.608	-0.5	.108
150°	-0.866	0.134	2.39	.880	-0.866	.014
180°	-1.0	0.0	2.72	1.00	-1.00	.000

u	$P_{\parallel}/P_{\perp}$	$P_{\parallel}$	$P_{\perp}$	$P_{\parallel} + P_{\perp}$
0°	0.57	0.568	1.0	1.57
30°	.55	.510	.933	1.434
60°	.48	.362	0.750	1.112
90°	.37	.184	.500	.684
120°	.22	.054	0.250	.304
150°	.104	.007	0.065	.072
180°	0	.000	0.000	0.000


W
0
0.134
0.5
1.0
1.5
1.866
2.0
P_0
1.0
0.8
0.6
0.4
0.2
0
$P_\perp$
$P_{\parallel}$
$P_{\parallel} + P_\perp$
2.0
1.0
0
$B = B_0 e^{-\alpha(\cos u - 1)}$
$P_\perp = \frac{P_0}{2}(1 + \cos u) = \frac{P_0}{2} W$
$P_{\parallel} = \frac{P_0}{2}\left\{W - \frac{1}{\alpha}\left[1 - e^{-\alpha W}\right]\right\}$
$W = (1 + \cos u)$
B_0
B
$\alpha = 1$
$R = \frac{B(\pi)}{B(o)}$
$= e^{2\alpha} = 7.4$
$\frac{P_{\parallel}}{P_\perp}$
0.6
0.5
0.4
0.3
0.2
0.1
0

for them to exist. The condition is[1]

$$P_\perp\left(\frac{P_\perp}{P_\parallel} - 1\right) < \frac{B^2}{8\pi} \quad \text{stable}, \qquad P_\perp > P_\parallel \tag{8}$$

This can be rewritten as

$$\beta < \left(\frac{P_\perp}{P_\parallel} - 1\right)^{-1} \qquad \beta = \frac{P_\perp}{B^2/8\pi} \tag{9}$$

Using the solution found earlier, this requires that

$$\beta < \frac{\alpha w - [1 - e^{-\alpha w}]}{[1 - e^{-\alpha w}]} \tag{10}$$

First consider this condition as evaluated at the center of the machine ($w = 2$). This yields

$$\beta < \left\{\frac{R}{R-1} \log R - 1\right\} \tag{11}$$

For a mirror ratio $R = 2$, this yields $\beta < 0.38$. For $R = 4$, $\beta < 0.84$. These are very unrestrictive conditions, and are probably too high to be valid for the low β solutions from which they were derived.

Near the ends, $P_\perp/P_\parallel \to \infty$, and one might be concerned about the restriction implied by this. However, because of the variation of β with u, this turns out to be of no concern, since in the limit $w \to 0$, we find

$$\beta_0 < \alpha e^{2\alpha}\left\{1 + \frac{\alpha w}{6} + \dots\right\} \tag{12}$$

where β_0 is the value of β at the center of the machine.

$$\beta_0 < \frac{R}{2} \log R \tag{13}$$

For $R = 2$, $\beta_0 < 0.69$, this is thus a less restrictive condition than (11).

We conclude that the type of velocity space instability here considered should not be of concern in the mirror machine for β values of present interest, as long as the fields and pressure variations in the vicinity of the mirrors are reasonably well approximated by the functional forms here considered.

Solutions to Axial Pressure Balance

Take $B(u) = B_0(1 - \alpha \cos u)$ See Fig. 2.

$$\therefore \nabla B = B_0\, \alpha \sin u$$

$$\frac{\nabla B}{B} = \frac{\alpha \sin u}{(1 - \alpha \cos u)}$$

$$\approx \alpha \sin u \quad \text{for} \quad \alpha << 1$$

1. S. Chandrasekhar, A. Kaufman, and K. Watson, Proc. Roy. Soc. A245, 435 (1958).

To solve:

$$\nabla P_{\parallel} = (P_{\parallel} - P_{\perp}) \frac{\nabla B}{B}$$

In the limit $\alpha << 1$

$$\frac{dP_{\parallel}}{du} = (P_{\parallel} - P_{\perp})\, \alpha \sin u\,, \qquad \text{take} \qquad P_{\perp} = \frac{P_0}{2}(1 + \cos u)$$

$$\left[P_{\perp}(\pi) = 0\right]$$

$$\therefore \quad \frac{dP_{\parallel}}{du} - P_{\parallel}\, \alpha \sin u = -\alpha \frac{P_0}{2} \sin u\,(1 + \cos u)$$

Solution: $$P_{\parallel}\, e^{-\alpha \int \sin u\, du} = -\int \frac{\alpha P_0}{2} \sin u\,(1 + \cos u)\, e^{-\alpha \int \sin u\, du}\, du + C$$

i.e., $$P_{\parallel}\, e^{\alpha \cos u} = -\frac{\alpha P_0}{2} \int (\sin u + \sin u \cos u)\, e^{\alpha \cos u}\, du + C$$

Let $\cos u = x$

$$P_{\parallel}\, e^{\alpha x} = +\frac{\alpha P_0}{2} \int (1 + x) e^{\alpha x}\, dx + C$$

$$= +\frac{\alpha P_0}{2} \left\{ \frac{e^{\alpha x}}{\alpha} + \left[\frac{x}{\alpha} - \frac{1}{\alpha^2}\right] e^{\alpha x} \right\} + C$$

Now $P_{\parallel} = 0$ at $u = \pm \pi$, i.e., at $x = -1$

$$\therefore \quad 0 = +\frac{\alpha P_0}{2} \left\{ \frac{e^{-\alpha}}{\alpha} + \left[-\frac{1}{\alpha} - \frac{1}{\alpha^2}\right] e^{-\alpha} \right\} + C$$

$$= +\frac{P_0}{2} \left\{ 1 - 1 - \frac{1}{\alpha} \right\} e^{-\alpha} + C$$

$$\therefore \quad c = +\frac{P_0}{2\alpha} e^{-\alpha}$$

$$\therefore \quad P_{\parallel}\, e^{\alpha x} = +\frac{P_0}{2} \left\{ e^{\alpha x} + \left[x - \frac{1}{\alpha}\right] e^{\alpha x} + \frac{e^{-\alpha}}{\alpha} \right\}$$

$$P_{\parallel} = -\frac{P_0}{2} \left\{ \frac{1}{\alpha}\left[1 - e^{-\alpha(x+1)}\right] - [x + 1] \right\}$$

at $u = 0$ $(x = 1)$,

$$P_{\parallel} = -\frac{P_0}{2} \left\{ \frac{1}{\alpha}\left[1 - e^{-2\alpha}\right] - 2 \right\}$$

$$= \frac{P_0}{2}\left\{2 - \frac{1}{a}\left[1 - e^{-2a}\right]\right\}$$

$$\approx P_0\, a \qquad (a << 1)$$

$$\boxed{P_{\parallel} = \frac{P_0}{2}\left\{[x+1] - \frac{1}{a}\left[1 - e^{-a(x+1)}\right]\right\}} \qquad x = \cos u$$

In terms of u :

$$P_{\parallel} = \frac{P_0}{2}\left\{[\cos u + 1] - \frac{1}{a}\left[1 - e^{-a(\cos u + 1)}\right]\right\}$$

$$\frac{dP_{\parallel}}{du} = \frac{P_0}{2}\left\{-\sin u - \left[(-\sin u)\, e^{-a(\cos u + 1)}\right]\right\}$$

$$= \frac{P_0}{2}\sin u\left\{e^{-a(\cos u + 1)} - 1\right\}$$

$$\therefore \text{ at } u = 0, \ \pm\pi, \quad \frac{dP_{\parallel}}{du} = 0$$

Also $$\frac{P_{\parallel}}{P_{\perp}} = \left\{1 - \frac{1}{a}\left[\frac{1 - e^{-a(1+\cos u)}}{1 + \cos u}\right]\right\}$$

Let $$w = 1 + \cos u \quad \begin{Bmatrix} w = 0 & \text{at} & u = \pm\pi \\ w = 2 & \text{at} & u = 0 \end{Bmatrix}$$

$$\therefore \quad \frac{P_{\parallel}}{P_{\perp}} = \left\{1 - \frac{1}{a}\left[\frac{1 - e^{-aw}}{w}\right]\right\}$$

$$\frac{P_{\parallel}}{P_{\perp}} \approx \left\{\frac{aw}{2} - \ldots\right\} = \left\{\frac{a}{2}(1 + \cos u) - \ldots\right\}$$

$$aw << 1$$

Returning to original assumptions, if $\nabla B/B = a \sin u$, exactly, then

$B = B_0\, e^{-a\cos u}$ is the resulting form for B.

The mirror ratio $$R = \frac{B_{max}}{B_{min}} = e^{2a}$$

The solutions found before now apply exactly.

$$P_{\parallel} = \frac{P_0}{2}\left\{[\cos u + 1] - \frac{1}{a}\left[1 - e^{-a(\cos u + 1)}\right]\right\}$$

$$\boxed{\begin{aligned} P_{\parallel} &= \frac{P_0}{2}\left\{w - \frac{1}{a}\left[1 - e^{-aw}\right]\right\} \\ P_{\perp} &= \frac{P_0}{2}(1 + \cos u) = \frac{P_0}{2}\, w \end{aligned}}$$

$$\frac{P_{\parallel}}{P_{\perp}} = \left\{1 - \frac{1}{a}\left[\frac{1 - e^{-aw}}{w}\right]\right\} \to 1 \quad \text{as} \quad a \to \infty \quad \text{(for } w \text{ between 0 and 2)}$$

$$P_{\parallel} - P_{\perp} = -\frac{P_0}{2a}\left\{1 - e^{-aw}\right\}$$

$$\frac{dP_{\parallel}}{du} = \frac{P_0}{2}\sin u\left\{e^{-a(\cos u + 1)} - 1\right\}$$

$$= \frac{P_0}{2}\, w^{1/2}(2 - w)^{1/2}\left\{e^{-aw} - 1\right\}$$

$$\frac{\nabla B}{B} = a \sin u = a\, w^{1/2}(2 - w)^{1/2}$$

Check: $\quad \nabla P_{\parallel} = (P_{\parallel} - P_{\perp})\dfrac{\nabla B}{B}$

$$\frac{P_0}{2}\, w^{1/2}(2 - w)^{1/2}\left\{e^{-aw} - 1\right\} \overset{?}{=} -\frac{P_0}{2a}\left\{1 - e^{-aw}\right\} a w^{1/2}(2 - w)^{1/2}$$

Also, $$\frac{\nabla P_{\parallel}}{P_{\parallel}} = \frac{w^{1/2}(2 - w)^{1/2}\left[e^{-aw} - 1\right]}{w - \frac{1}{a}\left[1 - e^{-aw}\right]}$$

$$= \frac{-a w^{1/2}(2 - w)^{1/2}}{\dfrac{aw}{1 - e^{-aw}} - 1}$$

$$\to \frac{-2(2 - w)^{1/2}\left(1 - \frac{aw}{2}\right)}{w^{1/2}} \to \frac{2}{w^{1/2}}\left(\frac{dw}{du}\right) \quad \text{as} \quad w \to 0$$

$$B = B_0\, e^{-a\cos u}$$

$$P_{\perp} = \frac{P_0}{2}(1 + \cos u) \qquad P_{\parallel} = \frac{P_0}{2}\left\{(1 + \cos u) - \frac{1}{a}\left[1 - e^{-a(1 + \cos u)}\right]\right\}$$

SOME HYDROMAGNETIC EQUILIBRIA

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Abstract

Hydromagnetic equilibria have been obtained for a variety of situations which differ little from that of a zero pressure uniform axial magnetic field. The perturbations considered are particle pressure, axial current, curvature of the system, and multipolar fields. These equilibria differ from those which have been obtained previously in the thermonuclear program in that the lowest order term in an asymptotic expansion of the magnetic surface is not cylindrically symmetric but is a function of both r and θ.

The problem is reduced to the solution of a second order nonlinear partial differential equation. If it is assumed that the lowest order terms in the expressions for the material pressure and axial current distributions are of the form $a + b\,\Psi_0$ where a and b are constants and Ψ_0 is the zeroth order magnetic surface, the equation is linear and can be integrated directly.

Our interest in this problem arose several years ago when the question of how the stellarator can be made hydromagnetically stable was being studied. In that study an asymptotic expansion was made in which the differences between the equilibria and an infinitely long cylindrical system with a uniform axial magnetic field are small. It was shown,[1] in particular, that the system is stable for a system with a multipolar field which depends on θ and z as $\sin(\ell\theta - hz)$ if $\beta < \delta^2$ where β is a measure of the material pressure and δ the strength of the multipolar field.

Efforts are being made to extend that theory in several directions. For example, the perturbations which limit the stability were found to be localized. The assumption that the Larmor radius is small compared to

1. J. L. Johnson, C. R. Oberman, R. M. Kulsrud, and E. A. Frieman, Phys. Fluids 1, 281 (1958).

* On loan from Westinghouse Electric Corp. Atomic Power Department.

the distance over which the perturbation can change should be removed. Also, only equilibria in which the lowest order term in an expansion of the magnetic surfaces[2] was cylindrically symmetric were considered. In this case the perturbations could be Fourier analyzed in θ and the modes separated. In more general equilibria which do not have this symmetry, for example, toroidal systems, the modes are coupled. In this work we carry through the first step of the stability problem for these systems; we develop a way of determining the equilibria.

The conditions which must be satisfied for an equilibrium to exist are[2]:

$$\nabla p = \underline{j} \times \underline{B} \,, \tag{1}$$

$$\nabla \times \underline{B} = 4\pi \underline{j} \,, \tag{2}$$

$$\nabla \cdot \underline{B} = 0 \,. \tag{3}$$

The magnetic surfaces must satisfy the condition

$$B \cdot \nabla \Psi = 0 \,. \tag{4}$$

It immediately follows from Eqs. 1 and 4 that p is a function of Ψ alone. We prescribe a magnetic field which satisfies these equations and then determine the magnetic surfaces which exist in this field. In order to avoid the complications due to boundary conditions at infinity we assume that a perfectly conducting wall is placed on one of these magnetic surfaces.

In order to treat toroidal systems we work in a coordinate system where the element of arc is

$$ds^2 = dr^2 + r^2 d\theta^2 + (1 - \kappa r \cos\theta)^2 dz^2 . \tag{5}$$

We use the usual thin tube limit and consider $\kappa a << 1$ where a is the average radius of the perfectly conducting wall. We separate our plasma currents into two terms. The component of $\underline{J}$ perpendicular to $\underline{B}$ is the diamagnetic current related to the presence of a material pressure and will in lowest order be denoted by the parameter β. We introduce the parameter $\Sigma = \underline{J} \cdot \underline{B}/\underline{B} \cdot \underline{B}$ to denote the current along the magnetic lines of force. It can easily be shown that

$$\underline{B} \cdot \nabla \Sigma = \nabla(\underline{B} \cdot \underline{B}) \times \underline{B} \cdot \nabla p/(\underline{B} \cdot \underline{B})^2 \,. \tag{6}$$

Finally multipolar fields[1] due to currents in external helical conductors are denoted by the parameter δ. As in reference 1, we make the parameters κ, β, Σ, and δ small and, in order that the effect of the parameters enters into the determination of the shape of the magnetic surfaces, we order them so that $\kappa \sim \beta \sim \Sigma \sim \delta^2 \sim \lambda^2$ where λ is an arbitrary expansion parameter.

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2. L. Spitzer, Jr., Phys. Fluids 1, 253 (1958).

We consider the magnetic field

$$\underline{B} = \underline{B}_o + \underline{B}_\kappa + \underline{B}_\beta + \underline{B}_\Sigma + \underline{B}_\delta + \ldots , \tag{7}$$

where

$$\underline{B}_o = \underline{e}_z B_o \text{ (a constant)},$$

$$\underline{B}_\kappa = \underline{e}_z B_o \kappa r \cos\theta ,$$

$$\underline{B}_\beta = \underline{e}_z B_\beta (\Psi_o) ,$$

$$\underline{B}_\Sigma = \nabla \times \underline{e}_z A_\Sigma ,$$

$$\underline{B}_\delta = \nabla \sum_{s>0} \sum_{\ell=-\infty}^{\infty} (1/s\kappa)\, \mathcal{E}_{\delta;\ell s} B_o I_\ell(x_s) \sin u_{\ell s} ,$$

and

$$x_s = s\kappa r ,$$

$$u_{\ell s} = \ell\theta - s\kappa z + \phi_{\ell s} .$$

Here $B_\beta(\Psi_o)$ is arbitrary, and the vector potential $\underline{A}_\Sigma$ must satisfy

$$\nabla \cdot \nabla \underline{A}_\Sigma = 4\pi \Sigma \underline{B}_o . \tag{8}$$

This system is more general than the one considered in reference 1 due to the presence of $\underline{B}_\kappa$ and the possibility that two $\mathcal{E}_{\delta;\ell s}$ with the same value of s but different ℓ's can exist.

When we carry out our expansion, Eq. 4 becomes

$$\sum_{n=0}^{m} \underline{B}_n \cdot \nabla \Psi_{m-n} = 0 . \qquad (m = 0, 1, 2, \ldots) \tag{9}$$

The condition that Ψ be periodic over the length $2\pi/\kappa$ is

$$\int_0^{2\pi/\kappa} \sum_{n=1}^{m} \underline{B}_n \cdot \nabla \Psi_{m-n}\, dz = 0. \qquad (m = 0, 1, 2 \ldots) \tag{10}$$

A similar set of conditions can be obtained for Σ_n. In the zeroth order Eq. 9 requires that Ψ_o be a function of r and θ alone. In the first order Ψ_λ is determined up to an arbitrary function of Ψ_o. In the second order Eq. 10 limits Ψ_o so that, if $u_{\ell m} = u_{\ell s} - u_{ms}$,

$$\Psi_o = \frac{2\pi}{\kappa}\Big\{ \sum_{\ell, m, s} \frac{B_o}{2} \mathcal{E}_{\delta;\ell s} \mathcal{E}_{\delta;ms} \frac{mr}{x_s^2} I_\ell'(x_s) I_m(x_s) \cos u_{\ell m} - A_\Sigma\Big\}. \tag{11}$$

The other set of equations limits the lowest order term in Σ,

$$\sum_{\lambda\lambda} = \frac{2\pi p_\beta'(\Psi_o)}{\kappa B_o}\Big\{-\frac{1}{2}\sum_{\ell, m, s} \mathcal{E}_{\delta;\ell s} \mathcal{E}_{\delta;ms} [I_\ell' I_m' + \frac{\ell m + x_s^2}{x_s^2} I_\ell I_m] \cdot$$

$$\cos u_{\ell m} + 2\kappa r \cos\theta\Big\} + g(\Psi_o) . \tag{12}$$

The problem then is to solve Eqs. 8, 11, and 12 simultaneously. In the particular case where p_β and g are linear in Ψ_o we can carry through the integration explicitly.

We will illustrate how the magnetic surfaces are distorted for a few special cases. Since we are quite familiar with multipolar fields, we will first limit ourselves to systems with $\kappa = p_\beta = \Sigma_{\lambda\lambda} = 0$, and with two such fields present.

In Fig. 1 $\ell = 1$, $m = 0$, and $\alpha = \mathcal{E}_{\delta;ms}/\mathcal{E}_{\delta;\ell s} = 0.7$. An $m = 0$ field, of course, is just a bulge. The surfaces are still basically circular although the magnetic axis has been shifted away from the center of the system. As α is increased the position of this fixed point is moved outward, so that it is infinitely far out when $\alpha = 1$. From then on the surfaces are open.

The case where $\ell = 2$, $m = 0$, $\alpha = 0.65$ is shown in Fig. 2. Outside the fixed points located at $x = 3$, the surfaces are ellipses and, for large x nearly circular. If α is less than 0.5, these fixed points are at the origin and all the surfaces are ellipses. If α is greater than 1.0, the fixed points are at infinity and all the surfaces are open.

Figure 3 shows the surfaces if $\ell = 3$, $m = 0$, $\alpha = 0.01$. The ellipsoidal surfaces around the three fixed points exist no matter how small α is. They go to infinity as α goes to 1 so that no closed surfaces then exist.

The situation when $\ell = 3$, $m = 2$, $\alpha = 0.1$ is shown in Fig. 4. The fixed point which is at $x = 4$ goes to infinity as α goes to 1.0 and the one at $x = 2$ goes to infinity as α goes to 1.5. For $\alpha < 1.0$ and $\alpha > 1.5$ then the surfaces are all closed. For α in this range they are closed for small x and open for large x .

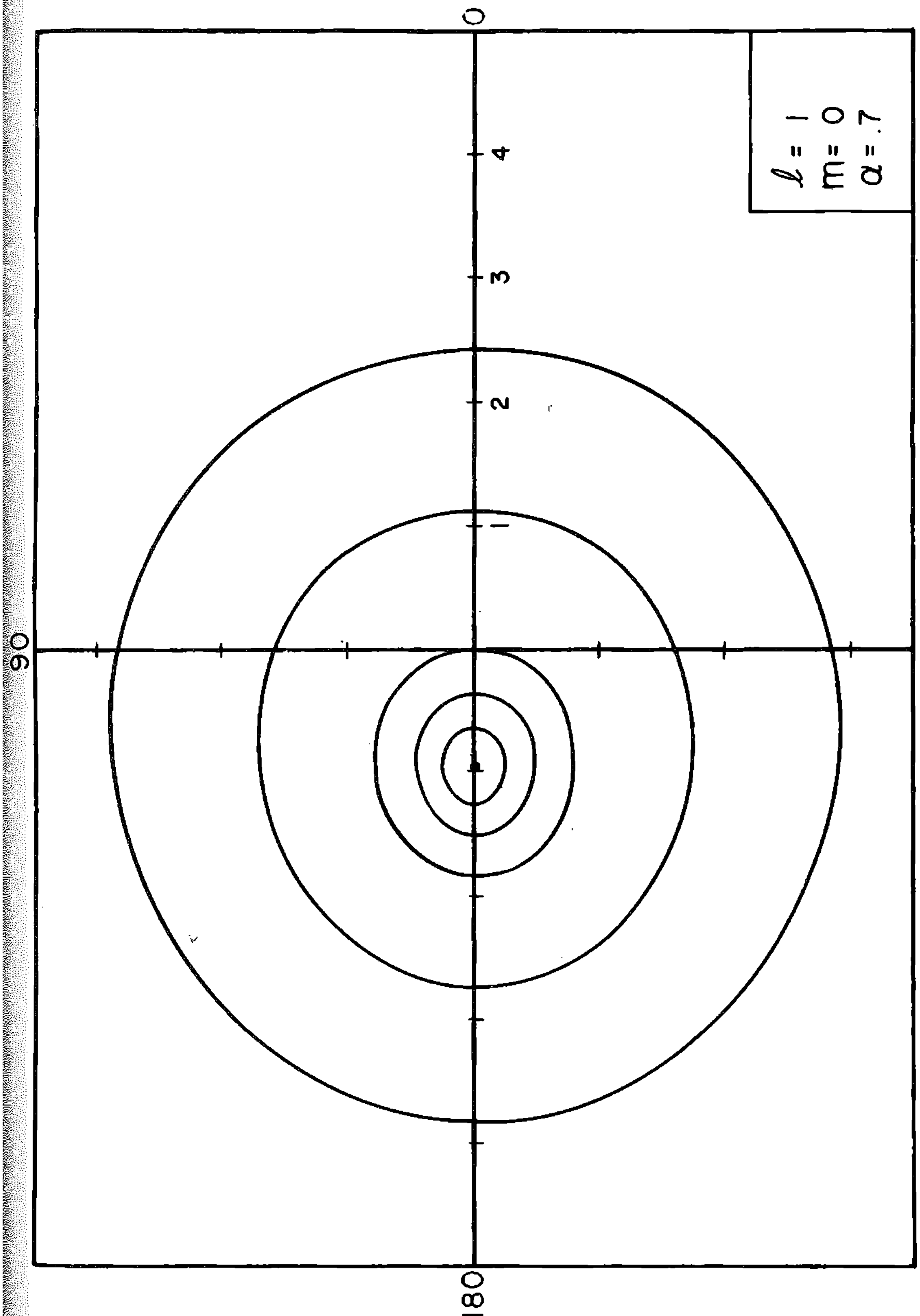

Figure 1

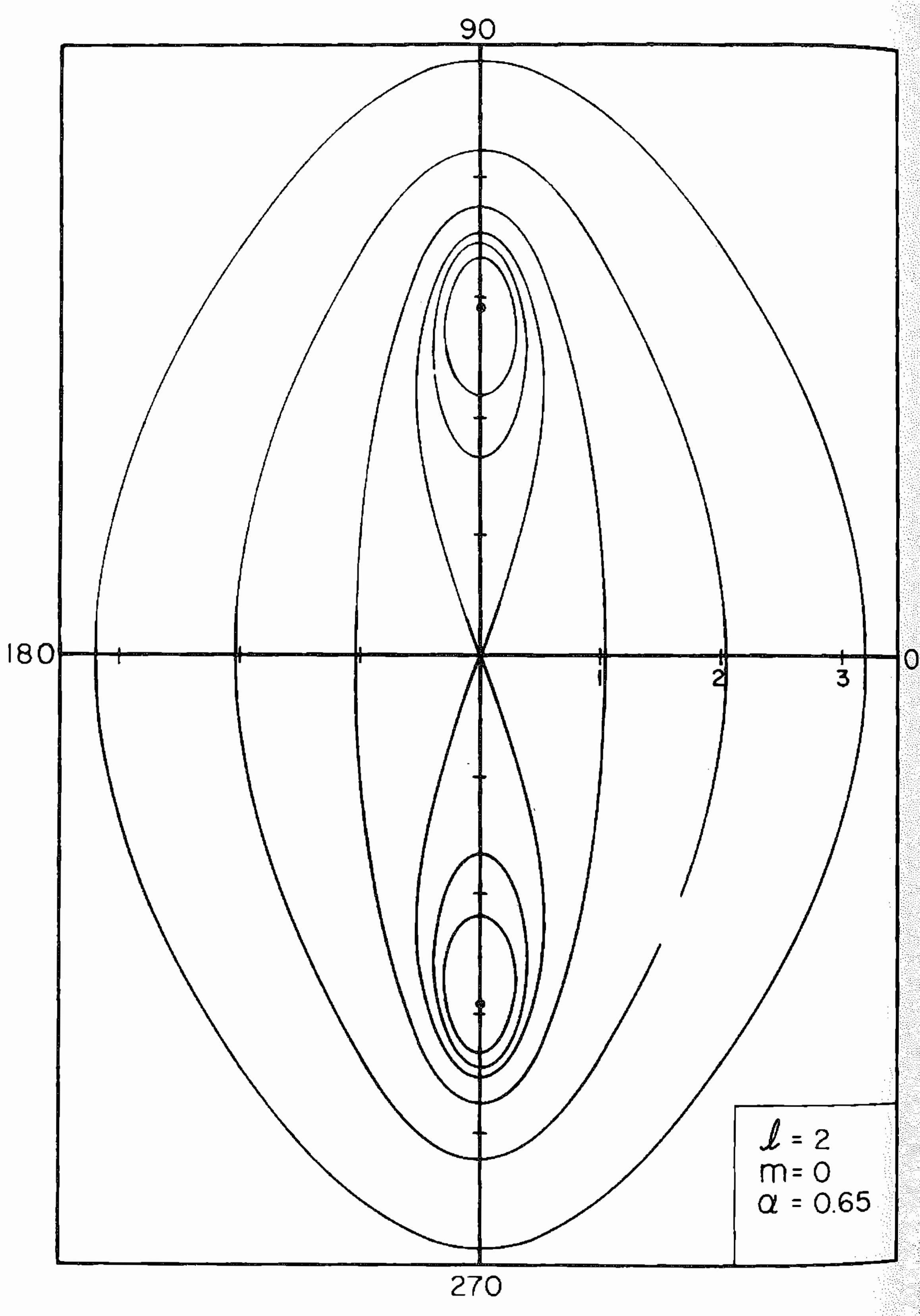

Figure 2

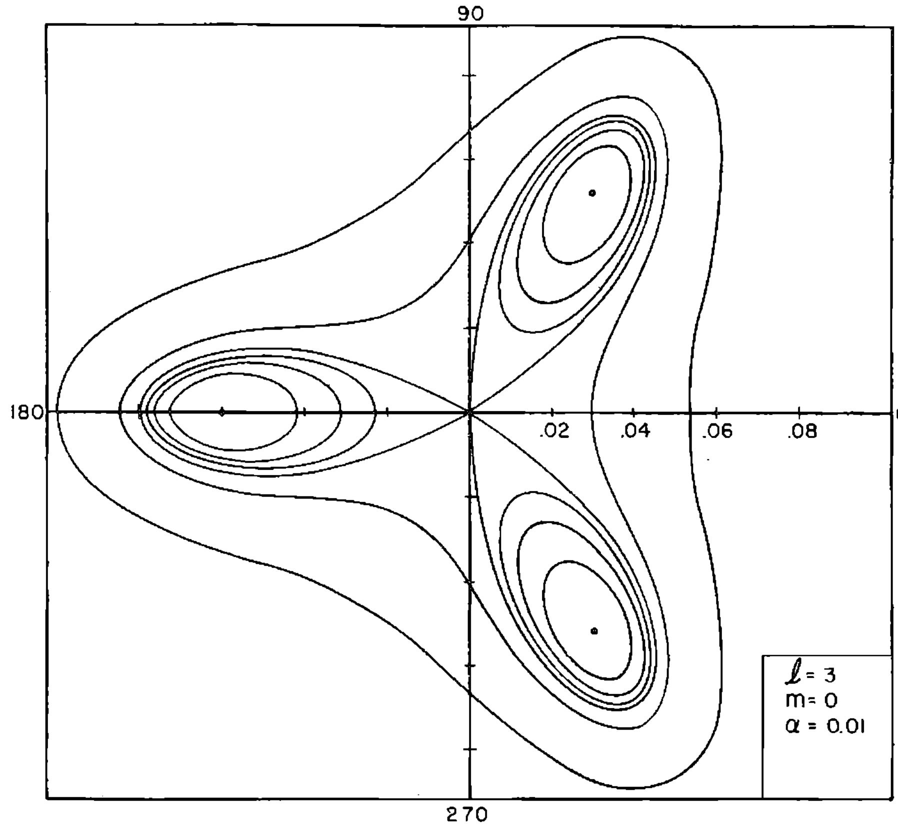

Figure 3

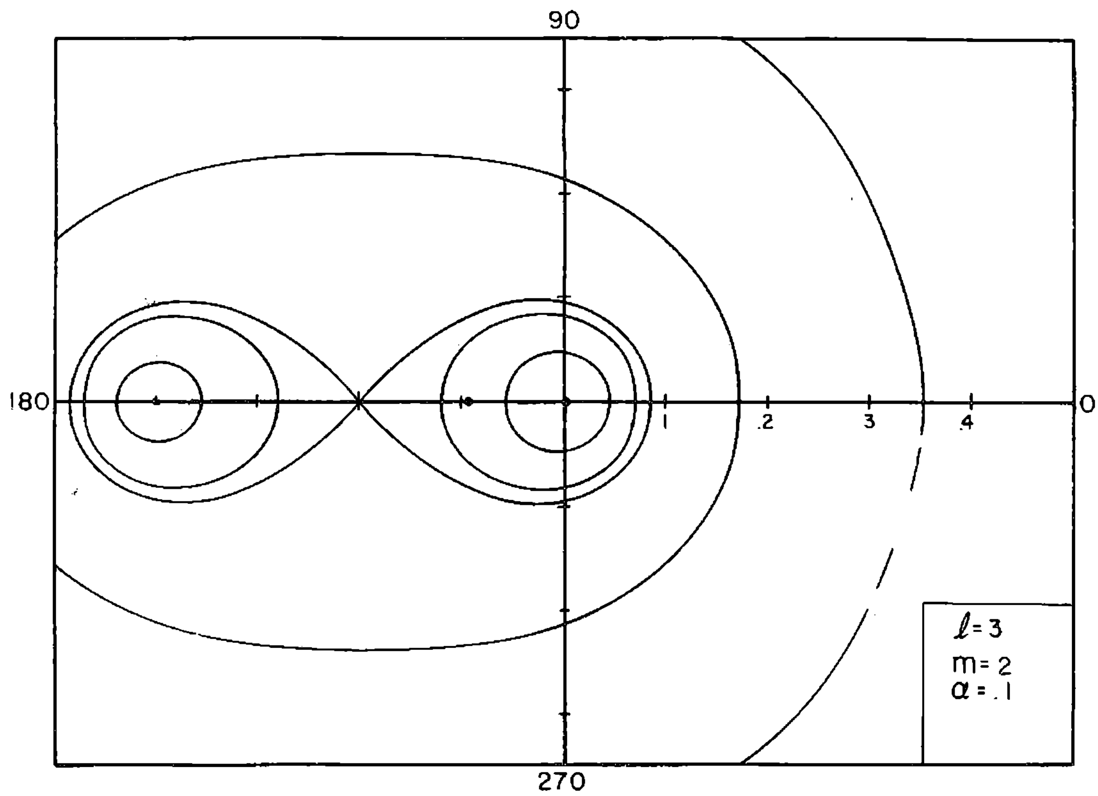

Figure 4

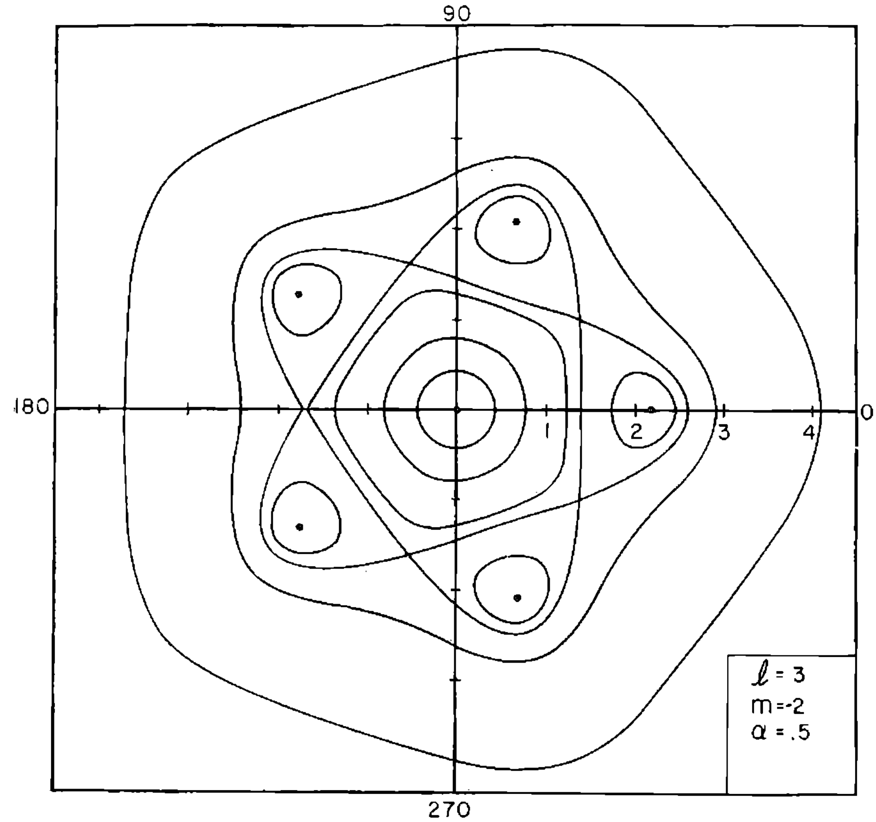

Figure 5

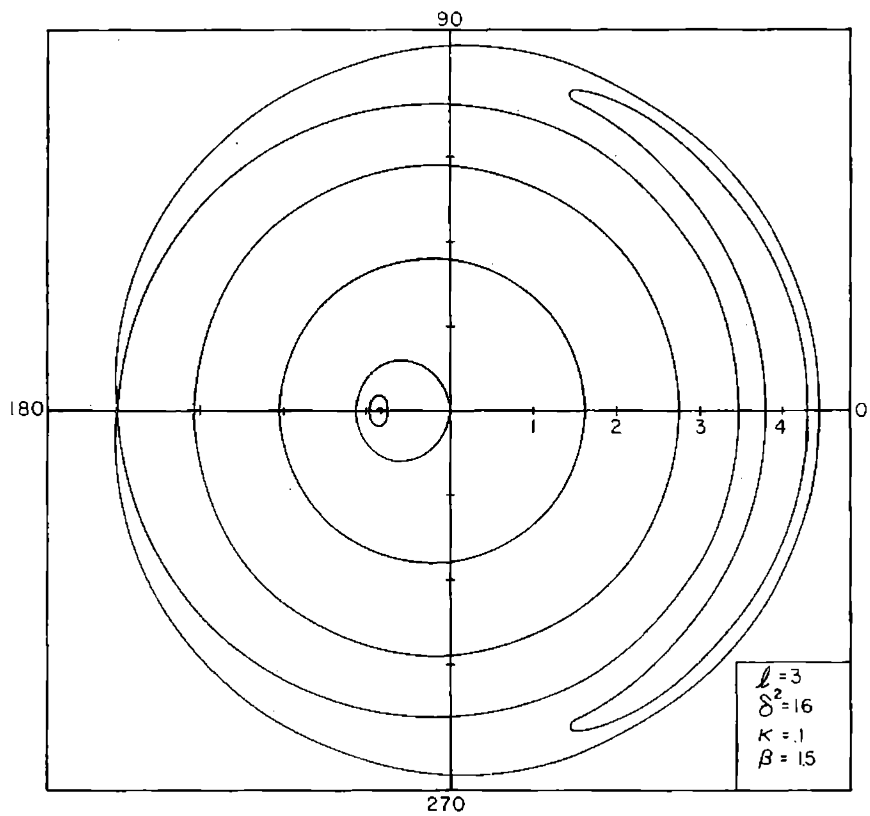

Figure 6

A rather pretty case is obtained when $\ell = 3$, $m = -2$, $\alpha = 0.5$ as shown in Fig. 5. Again open surfaces exist for large x if $1.0 < \alpha < 1.5$.

Finally, if only one multipolar field with $\ell = 3$ is present in a torus in which material is present, the magnetic surfaces are as shown in Fig. 6. The numbers were selected solely to illustrate how the surfaces are distorted. We note in particular that the magnetic axis is displaced outward, away from the center of the torus.

We have been able to identify Ψ_0 with the magnetic flux through a ribbon which has one side on the geometric axis of the system and the other side embedded in the surface in a constant θ plane. By also calculating the flux through a constant z cross section of a Ψ_0 surface we can get the rotational transform.

We are indebted to other members of the Matterhorn theoretical group, particularly to Martin Kruskal, for many helpful discussions.

SOME AXIALLY SYMMETRIC PROBLEMS IN MAGNETO-HYDRODYNAMICS*

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Abstract

We carry out the details for solving certain boundary value problems for $\nabla p = J \times B$ considered by Grad and Rubin. We show how the given data allow a reduction to the Dirichlet problem for a non-linear elliptic equation. The method of iterations is used to solve the problem in small domains.

1. Introduction

In one of their Geneva papers[1] Grad and Rubin considered certain boundary value problems for the system

$$\nabla p = J \times B \tag{1.1}$$

$$\nabla \times B = \mu J \tag{1.2}$$

$$\nabla \cdot B = 0 \tag{1.3}$$

in tubular volumes. (Here p is fluid pressure, J the current density, and B the magnetic field.) For instance, in a volume V pictured in Fig. 1, suppose that B_n (the inward normal component of B_n) is given on the entire surface in such a way that $B_n > 0$ on S_1 , $B_n < 0$ on S_2 and $B_n = 0$ on S_3. In addition, p and J_n are given on S_1. Grad and Rubin gave many arguments to show that this and similar problems are well posed (i.e., that one can solve for p, J, and B is V). It is the purpose of this

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1. H. Grad and H. Rubin, Hydrodynamic equilibria and force-free fields, Proc. of 2nd U. N. International Conference on the Peaceful Uses of Atomic Energy, Sept. 1958, Vol. 31.

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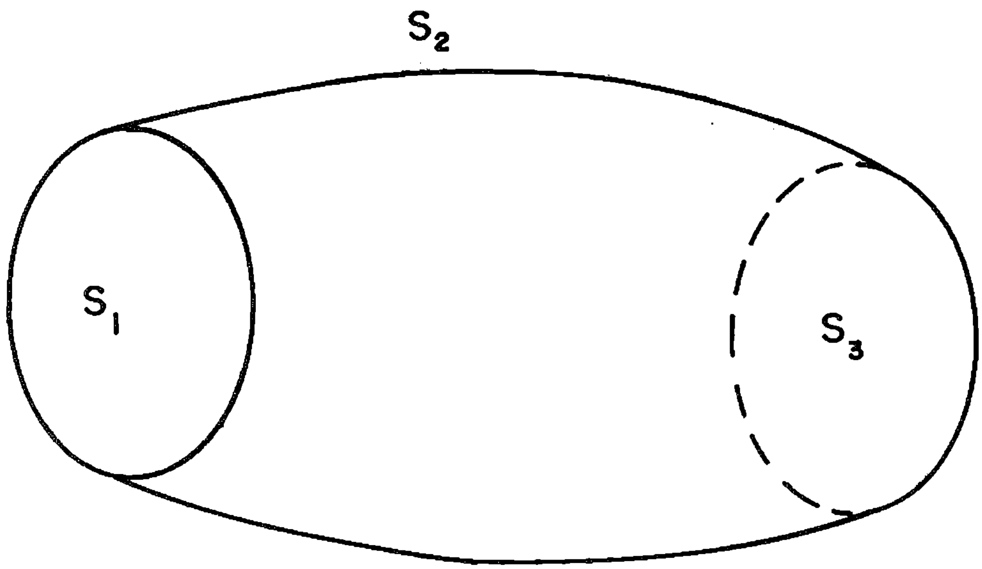

Fig. 1

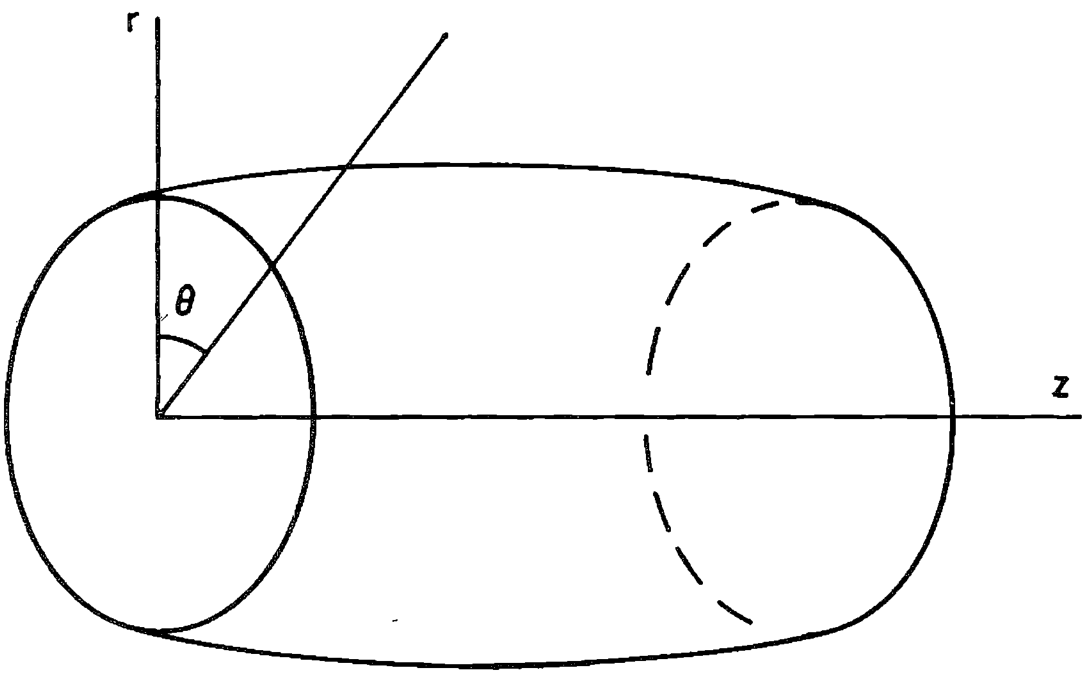

Fig. 2

paper to show how one can solve the problem in the case that V and all given quantities are axially symmetric (i.e., do not depend on θ if we introduce the coordinate system r, z, θ as shown in Fig. 2). In this case many simplifications can be made and mathematically the problem can be reduced to two dimensions (as observed in Ref. 1). Indeed, most of our methods are contained implicitly in Ref. 1, only they do not carry out the details.

We should remark that there is no difference whether V has a hole through it (as in Fig. 3) or not; the mathematical treatment is identical. V may even be a torus about the z axis.

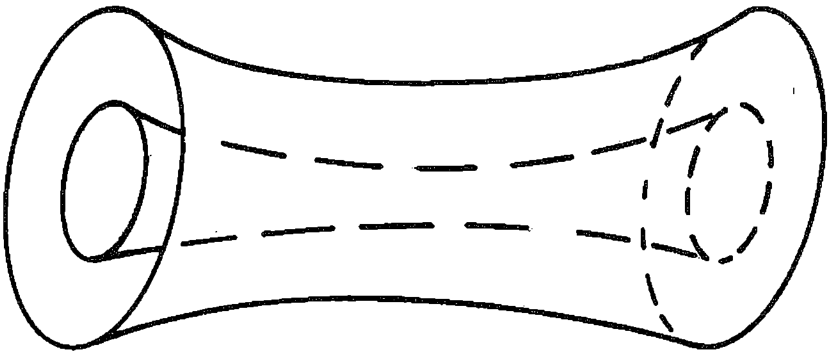

Fig. 3

Also, it should be noted that p on J_n may be prescribed on S_2 instead of S_1. Moreover, in place of J_n we may prescribe the "twist" of the B lines on each tubular p surface.

The author would like to thank Professor H. Grad for his encouragement.

2. Mathematical Formulation.

If we introduce the coordinates indicated in Fig. 2 and set* $u = B_r$, $v = -B_z$, $w = B_\theta$, Eqs. 1.1 - 1.3 become

$$\mu p_r + v(u_z + v_r) + \frac{1}{r}(rw)_r = 0 \qquad (2.1)$$

$$\mu p_z + u(u_z + v_r) + w\, w_z = 0 \qquad (2.2)$$

$$u(wr)_r - v(wr)_z = 0 \qquad (2.3)$$

$$(ur)_r - (ur)_z = 0 \qquad (2.4)$$

---

* When the subscripts r, z, θ appear on a capital letter, they denote components of a vector; when they appear on a small letter, they denote partial differentiation.

Let G be the two dimensional domain obtained from V by slicing it with the plane $\theta = 0$. We only consider the portion $r > 0$. (If there is no hole through V, $r = 0$ is part of the boundary of G.) Let C_1, C_2, C_3 be the curves bounding G and corresponding to S_1, S_2, S_3, respectively. Call the remaining curve C_4. (Thus C_4 corresponds to the inner surface if there is a hole through V or is the line $r = 0$ if there is no hole through V.) Let $0 \leq s \leq s_0$ and $0 \leq \sigma \leq \sigma_0$ denote arc lengths along the curves C_1 and C_2 respectively, oriented in such a way that $s = 0$ at the

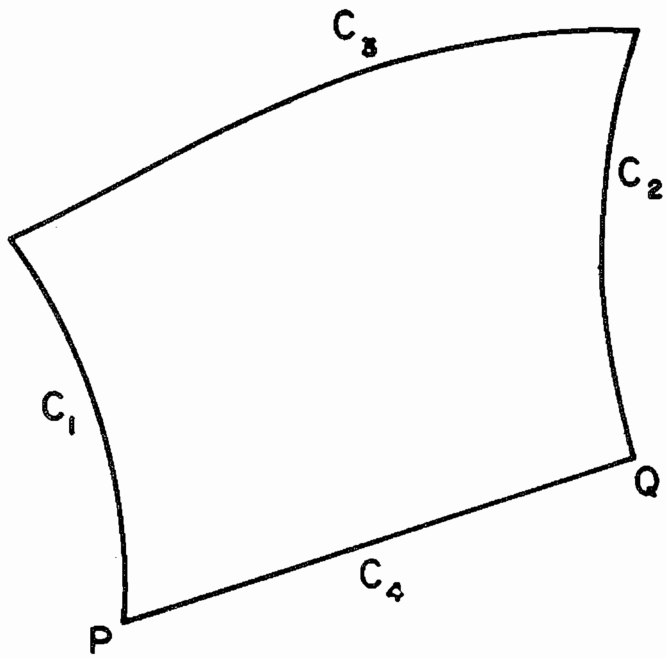

Fig. 4

intersection P of C_1 and C_4 and $\sigma = 0$ at the intersection Q of C_2 and C_4 (Fig. 4).

The given boundary conditions are

$$B_n = a(s) > 0 \qquad \text{on } C_1 \tag{2.5}$$

$$= -b(\sigma) < 0 \qquad \text{on } C_2 \tag{2.6}$$

$$= 0 \qquad \text{on } C_3 \text{ and } C_4 \tag{2.7}$$

$$p = c(s) \qquad \text{on } C_1 \ (\text{or } p = c_1(\sigma) \text{ on } C_2) \tag{2.8}$$

$$J_n = e(s) \qquad \text{on } C_1 \ (\text{or } J_n = e_1(\sigma) \text{ on } C_2) \tag{2.9}$$

$$B_\theta = \ell \qquad \text{at P} \ (\text{or } B_\theta = \ell_1 \text{ at Q}) \tag{2.10}$$

where $a(s)$, $b(\sigma)$, ... are given smooth functions. By the divergence theorem we must assume

$$\int_0^{s_0} a(s)\, r(s)\, ds = \int_0^{\sigma_0} b(\sigma)\, r(\sigma)\, d\sigma \equiv \nu , \tag{2.11}$$

where $r(s)$ and $r(\sigma)$ are the values of r at the points of C_1 and C_2 corresponding to the values of s and σ, respectively.

As shown in Ref. 1, we can reduce the system 2.1-2.4 still further. By 2.4, there is a function ψ (called a stream function for reasons which will become apparent) such that

$$\psi_z = ur \quad , \quad \psi_r = vr \quad , \; \psi(P) = 0 \tag{2.12}$$

Setting $q = rw$, we see from 2.3 that q is a function of ψ.*
Next, dividing 2.1 by v, 2.2 by u, and subtracting give

$$\begin{vmatrix} p_r & v \\ p_z & u \end{vmatrix} = 0$$

and hence p is also a function of ψ. Since $P_r = p'(\psi)\psi_r$,

$$u_z + v_r = \frac{1}{r}\left(\psi_{zz} + \psi_{rr} - \frac{1}{r}\psi_r\right) \equiv \frac{1}{r} L\psi \tag{2.13}$$

Hence 2.1 becomes

$$\mu \, v \, r \, p'(\psi) + \frac{1}{r} L\psi + \frac{1}{r^2} q(\psi)\, q'(\psi)\, vr = 0$$

or

$$L\psi + \mu r^2 p'(\psi) + q(\psi)\, q'(\psi) = 0 \tag{2.14}$$

Having reduced the system 2.1-2.4 to a single equation 2.14 for ψ, we now interpret the boundary conditions 2.5-2.7 in terms of ψ. On C_1

$$\psi_s = \psi_r r_s + \psi_z z_s = vrr_s + urz_s$$

$$= rB \cdot N = rB_n = r\, a(s)$$

where $N = (z_s \, , -r_s)$ is the inward drawn unit normal to C_1 (and hence to S_1). Hence

$$\psi = \int_0^s r(s)\, a(s) ds \equiv A(s) \qquad \text{on } C_1 \tag{2.15}$$

Similarly,

$$\psi = \int_0^\sigma r(\sigma)\, b(\sigma) d\sigma \equiv B(\sigma) \qquad \text{on } C_2 \tag{2.16}$$

Hence, the boundary conditions 2.5-2.7 can be written as

$$\psi = A(s) \qquad \text{on } C_1 \tag{2.17}$$

$$= B(\sigma) \qquad \text{on } C_2 \tag{2.18}$$

$$= \nu \qquad \text{on } C_3 \tag{2.19}$$

$$= 0 \qquad \text{on } C_4 \, . \tag{2.20}$$

---

* Here we tacitly assumed that $\psi_r^2 + \psi_z^2 = r^2(B_r^2 + B_z^2) \neq 0$. Only such flows are of interest physically.

At first glance, one might be led to believe that we have reduced our problem to solving the nonlinear partial differential equation 2.14 subject to the boundary conditions 2.17-2.20. However, it should be realized that we do not as yet know the functions $p(\psi)$ and $q(\psi)$ and until we can determine these functions we cannot hope to solve 2.14. We shall now show how the remaining boundary conditions determine $p(\psi)$ and $q(\psi)$ in the interval $0 \le \psi \le \nu$ (cf. 2.11).

Since $A'(s) = a(s) \neq 0$ in $0 \le s \le s_0$, we can solve $A(s) = A$ for s in the interval $0 \le A \le \nu$: $s = S(A)$. Thus

$S'(A) = \frac{1}{a(S(A))}$. Now on C_1

$$p = c(s) = c(S(A)) \qquad 0 \le A \le \nu$$

Hence, by 2.15

$$p(\psi) = c(S(\psi)) \qquad 0 \le \psi \le \nu \tag{2.21a}$$

Similarly,

$$\frac{dq}{ds} = q_r r_s + q_z z_s = \left((rw)_r r_s + rw_z z_s\right)$$
$$= -\mu r J_n .$$

Thus

$$q = -\mu \int_0^s r(s)\, e(s)\, ds + \ell r(0) \equiv E(s)$$

on C_1. Hence

$$q(\psi) = E(s(\psi)) \qquad 0 \le \psi \le \nu . \tag{2.21b}$$

It is convenient for purposes of solving 2.14 to extend the definitions of $p(\psi)$ and $q(\psi)$ outside the interval $0 \le \psi \le \nu$ to all values of ψ. This may be done in many ways. In particular, we may demand the following. Set

$$H(r,\psi) = \mu r^2 p'(\psi) + q(\psi) q'(\psi) \tag{2.22}$$

$$\lambda = \max_{\substack{r_1 \le r \le r_2 \\ 0 \le \psi \le \nu}} |H(r,\psi)| \tag{2.23}$$

$$M = \max_{\substack{r_1 \le r \le r_2 \\ 0 \le \psi \le \nu}} \left| \frac{\partial H}{\partial \psi}(r,\psi) \right| \tag{2.24}$$

where we assume that G is contained in the strip $r_1 \le r \le r_2$. Let $\varepsilon > 0$ be any fixed quantity. We continue the definition of $p(\psi)$ and $q(\psi)$ to the whole interval $-\infty < \psi < \infty$ in such a way that

$$\max_{\substack{r_1 \le r \le r_2 \\ -\infty < \psi < \infty}} |H(r,\psi)| - \lambda < \varepsilon \tag{2.25}$$

$$\max_{\substack{r_1 \le r \le r_2 \\ -\infty < \psi < \infty}} \left| \frac{\partial H}{\partial \psi}(r,\psi) \right| - M < \varepsilon \qquad (2.26)$$

3. Solving the Problem.

In this section we shall employ the method of iterations to solve 2.14 with the boundary conditions 2.17-2.20. (This is called a Dirichlet problem for 2.14.) The difficulty in solving it stems from the non-linearity of the term $H(r,\psi)$ (cf. 2.22). The advantage of our method is that it lends itself to calculation quite readily. At the end of this section we shall compare our results with those obtained using a powerful fixed point theorem of Berkhoff-Kellogg[2] and Schauder.[3] The drawback of the latter method is that it gives no hint as to how solutions may be calculated.

Let ψ_o be any smooth function in G satisfying the boundary conditions 2.17-2.20. The method of iterations demands that we be able to solve the linear equations

$$L\psi = - H(r,\psi_o)$$

with the boundary data 2.17-2.20. If G is bounded away from $r = 0$, the result is classical[4]. The case when G touches the z axis is treated in Appendix I of this paper and is taken from a more general result of the author's.[5]

We next form a sequence $\psi_o, \psi_1, \psi_2, \ldots$ where ψ_n is recursively defined as the solution of

$$L\psi_n = - H(r,\psi_{n-1})$$

satisfying the boundary conditions 2.17-2.20. Employing the norm

$$|| \Psi || = \max_{(r,z)\varepsilon G} | \Psi(r,z) | ,$$

we note that

$$| L(\psi_{n+1} - \psi_n) | \le (M + \varepsilon) || \psi_n - \psi_{n-1} ||$$

(cf. 2.26). Now assume that G is contained in the rectangle

$o \le r_1 \le r \le r_2$, $z_1 \le z \le z_2$. A simple application of the maximum principle shows that

$$|| \psi_{n+1} - \psi_n || \le \frac{(M+\varepsilon)}{4e} r_2^2 || \psi_n - \psi_{n-1} ||. \qquad (3.1)$$

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2. G. D. Birkhoff and O. D. Kellogg, Trans. Amer. Math. Soc., Vol. 23, 1922, pp. 96 - 115.

3. J. Schauder, Studia Math., Vol 2, 1930, pp. 171-180.

4. J. Schauder, Math. Zeit. Vol, 38, 1933-34, pp. 257 - 282.

5. M. Schechter, On the Dirichlet problem for second order elliptic equations with coefficients singular at the boundary, Communications on Pure and Applied Mathematics, to appear.

(The proof of 3.1 will be carried out in Appendix II.) Now assume that

$$r_2 < \sqrt{\frac{4e}{M}} . \qquad (3.2)$$

Then we can always find an $\varepsilon > 0$ such that $\textcircled{H} \equiv \frac{M+\varepsilon}{4e} r_2^{\,2} < 1$. This is a sufficient condition for the sequence $\psi_0, \psi_1, \psi_2, \ldots$ to converge uniformly to a limit function (cf. Appendix III). Moreover, it follows from the interior Schauder estimates (Ref. 4) that the limit function ψ has continuous second derivatives and satisfies

$$L\psi = - H(r,\psi) .$$

Since each of the functions ψ_n satisfies 2.17-2.20, the limit function ψ does likewise. Thus ψ is a solution to our problem.

A still easier argument shows that the iterations converge when 3.2 is replaced by

$$z_2 - z_1 < \sqrt{\frac{8}{M}} \qquad (3.3)$$

Since the quantity M plays such an important role in the method of iterations, we shall express it in a form in which the dependence of its magnitude upon physical quantities is apparent:

$$M = \max_{C_1} \left| \frac{r^2}{B_n} \frac{d}{ds} \left(\frac{\mu}{B_n} \frac{dp}{ds}\right) + \frac{1}{B_n} \frac{d}{ds} \left(\frac{B_\theta}{B_n} \frac{dB_\theta}{ds}\right) \right| \qquad (3.4)$$

We would like to mention that Bers and Nirenberg[6] were able to solve very general equations of the type considered here by making use of the fixed point theorem mentioned above. It follows from their work that a solution of 2.14, 2.17-2.20 exists when $r_1 > 0$ <u>without the restrictions</u> 3.2 or 3.3. Most likely their result can be carried over to the case $r_1 = 0$ without much difficulty.

4. Physically significant solutions.

In order to solve equation 2.14 we extended the definitions of $p(\psi)$ and $q(\psi)$ outside the interval $0 \leq \psi \leq \nu$. However, it is obvious that for a solution to have physical significance, it must be contained in that interval. In this section we shall give some configurations for which the solution of 2.14, 2.17-2.20 satisfies $0 \leq \psi \leq \nu$. Proofs are given in Appendix IV.

As before, we assume that G is contained in the rectangle $0 \leq r_1 \leq r \leq r_2$, $z_1 \leq z \leq z_2$. Define $H(r,\psi)$ and λ by 2.22 and 2.23, respectively.

Theorem 4.1. If $H(r,0) \geq 0$ and $H(r,\nu) \leq 0$ for all r in $r_1 < r < r_2$, then there is a solution ψ of 2.14, 2.17-2.20 satisfying $0 \leq \psi \leq \nu$ in G.

6. L. Bers and L. Nirenberg, On linear and non-linear elliptic boundary value problems on the plane, Atti del Convegno internazionale sulle Equazioni alle derivate parziali, Trieste, August 1954, pp 141 - 167.

Next assume that G is the rectangle $r_1 \leq r \leq r_2$, $z_1 \leq z \leq z_2$. If C_1 and C_2 are the lines $z = z_1$, $z = z_2$, respectively, then $s = \sigma = r - r_1$.

Theorem 4.2. If $r_1 > 0$ and there is an $\varepsilon > 0$ such that

$$\frac{\lambda + \varepsilon}{4} (2r^2 \log \frac{r}{r_1} + r_1{}^2 - r^2) \leq A(s), B(\sigma) \leq \nu - \frac{\lambda+\varepsilon}{4}(2r^2 \log \frac{r}{r_2} + r_2{}^2 - r^2),$$

then there is a solution ψ satisfying $o \leq \psi \leq \nu$.

For the special case when $J_n = 0$ on C_1, we can do slightly better. For then $|H(r,\psi)| \leq r^2 \lambda_1$, where λ_1 is some fixed constant. We do not have to assume $r_1 > 0$.

Theorem 4.3. If there is an $\varepsilon > o$ such that

$$\frac{\lambda_1 + \varepsilon}{8} (r^2 - r_1{}^2)^2 \leq A(s),\ B(\sigma) \leq \nu - \frac{\lambda_1+\varepsilon}{8} (r^2 - r_1{}^2)^2 \text{, then } 0 \leq \psi \leq \nu.$$

Finally, we mention the case when C_1 and C_2 are the lines $r = r_1$, $r = r_2$, respectively. Then $s = \sigma = z - z_1$.

Theorem 4.4. If there is an $\varepsilon > 0$ such that

$$\frac{\lambda+\varepsilon}{2} (z - z_1)^2 \leq A(s),\ B(\sigma) \leq \nu - \frac{\lambda+\varepsilon}{2} (z - z_2)^2 ,$$

then $0 \leq \psi \leq \nu$.

Appendix I.

The linear problem when $r_1 = 0$.

We wish to solve the Dirichlet problem for the equation

$$L\psi \equiv \psi_{rr} + \psi_{zz} - \frac{1}{r} \psi_r = f(r,z) \qquad (I.1)$$

where $f(r,z)$ does not depend on ψ. For $f = 0$, it was solved by Brousse and Poncin [7] and if one can exhibit a particular solution of I.1, their result gives the complete answer. Here we take another approach.

Let Ψ be any function which takes on the desired boundary values. Setting $\phi = \psi - \Psi$, we get as an equation for ϕ

$$L\phi = f(r,z) - L\,\Psi \equiv F(r,z) \qquad (I.2)$$

and $\phi = 0$ on $\mathring{G}$, the boundary of G. Hence if we can solve I.2 for any $F(r,z)$ with $\phi = 0$ on $\mathring{G}$, we can solve I.1 for any $f(r,z)$ and ψ having the desired boundary values. Thus we need only concentrate on the problem for ϕ.

7. P. Brousse and H. Poncin, Quelques resultas generaux concernant la determination de solutions d' equations elliptiques par les conditions aux frontiers, Jubile Scientifique de M. D. Riabouchinsky, Pub. Sci. et Tech. de Ministere de l' Air, Paris, 1954.

Lemma I.1. Suppose $|F(r,z)| \leq \lambda'$ and that ϕ is a solution of I.2 which vanishes on $\dot{G}$. Then

$$|\phi| \leq \frac{\lambda'}{2} r^2 \log \frac{r_2}{r} \equiv h(r) \tag{I.3}$$

in G.

Proof. It is easily checked that $Lh = -\lambda'$. Hence $L(\phi - h) = F + \lambda' \geq 0$ in G, while $\phi - h \leq 0$ on $\dot{G}$. Hence by the maximum principle [8,9] $\phi \leq h$ in G. The same argument for the function $-(\phi + h)$ gives the other half of the inequality I.3.

Returning to our problem, let G_n be the intersection of G with the halfplane $r > \frac{1}{n}$. Since G_n does not touch $r = 0$, we can solve I.2 in G_n with $\phi = 0$ on $\dot{G}_n$. Call the solution ϕ_n. If $n > m$,

we have, by Lemma I.1,

$$|\phi_n| \leq \frac{\lambda'}{2m^2} \log mr_2 \tag{I.4}$$

on $r = \frac{1}{m}$. Since $L(\phi_n - \phi_m) = 0$ in G_m, we have, by the maximum principle,

$$|\phi_n - \phi_m| \leq \frac{\lambda'}{2m} \log mr_2 \tag{I.5}$$

in G_m. If $\ell \leq m$, I.5 surely holds in G_ℓ. Now fix ℓ and let $m,n \longrightarrow \infty$. Then $|\phi_n - \phi_m| \longrightarrow 0$ uniformly in G_ℓ. Thus there is a continuous function ϕ in G_ℓ to which the ϕ_n converge. It follows from the interior Schauder estimates (Ref. 4) that ϕ has continuous second derivatives and that the derivatives of the ϕ_n converge to those of ϕ. Thus ϕ is a solution of I.2 in G_ℓ. Since ℓ was arbitrary, ϕ is a solution in G. That $\phi = 0$ on that part of $\dot{G}$ not touching $r = 0$ follows from the fact that each $\phi_n = 0$ there. That $\phi = 0$ on $r = 0$ follows from Lemma I.1.

Appendix II.

Proof of 3.1.

The proof of 3.1 follows almost immediately from Lemma I.1. Since

$$|L(\psi_{n+1} - \psi_n)| \leq (M + \varepsilon) \, \| \psi_n - \psi_{n-1} \| ,$$

we have, by Lemma 3.1,

$$|\psi_{n+1} - \psi_n| \leq \frac{(M+\varepsilon) \| \psi_n - \psi_{n+1} \|}{2} r^2 \log \frac{r_2}{r} .$$

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8. E. Hopf, Preuss. Akad. Wiss. Sitzungsber, Vol 19, 1927, pp. 147 - 152.

9. Bateman, Partial Differential Equations, Cambridge 1932, pp. 135 - 137.

Now the maximum of the function $r^2 \log \frac{r_2}{r}$ occurs at $r = e^{-1/2} r_2$ and equals $r_2^2/2e$. Hence

$$|\psi_{n+1} - \psi_n| \leq \frac{r_2^2}{4e} (M + \varepsilon) \| \psi_n - \psi_{n-1} \|$$

from which 3.1 is an immediate consequence.

Appendix III.

Proof of convergence of iterations.

If condition 3.2 is satisfied, then

$$\| \psi_{n+1} - \psi_n \| \leq (H) \| \psi_n - \psi_{n-1} \| \qquad \text{(III.1)}$$

where (H) is some constant < 1. Since III.1 holds for all n,

$$\| \psi_{t+1} - \psi_t \| \leq (H) \| \psi_t - \psi_{t-1} \| \leq (H)^2 \| \psi_{t-1} - \psi_{t-2} \| \leq \ldots$$

$$\leq (H)^t \| \psi_1 - \psi_o \| .$$

Thus for n > m

$$\| \psi_n - \psi_m \| \leq \sum_{t=m}^{n} \| \psi_{t+1} - \psi_t \| \leq \| \psi_1 - \psi_o \| \sum_{t=m}^{n} (H)^t$$

$$\leq (H)^m \| \psi_1 - \psi_o \| \cdot \frac{1}{1-(H)} \longrightarrow 0$$

as $m,n \rightarrow 0$. Thus ψ_o. ψ_1. ψ_2, ... forms a Cauchy sequence and there is a continuous function ψ to which they converge uniformly.

Appendix IV.

Proof of Theorem 4.1 - 4.4.

Proof of Theorem 4.1. Since $H(r,0) \geq 0$ for all r in $r_1 \leq r \leq r_2$, we may make $H(r,\psi) \geq 0$ for $\psi < 0$. Similarly we may assume $H(r,\psi) \leq 0$ for $\psi > \nu$. Now suppose ψ is a solution of

$$L\psi = - H(r,\psi)$$

satisfying the boundary conditions 2.17-2.20. If $\psi < 0$ anywhere in G, it must have a negative minimum at some interior point. At such a point $L\psi = -H(r,\psi) \leq 0$. Hence by the maximum principle, ψ must be identically a constant in the neighborhood of such a point. Thus the set of minimum points is open. By continuity it is closed. Hence ψ is identically a constant, which is impossible.

Proof of Theorem 4.2. Set

$$\phi = \psi + \frac{\lambda+\varepsilon}{4} \left(2r^2 \log \frac{r}{r_1} + r_1^2 - r^2 \right)$$

Then $L\phi = - H - (\lambda+\varepsilon) \leq 0$ in G, while $\phi \geq 0$ on $\dot{G}$. Hence $\phi \geq 0$ inside. Similar reasoning proves the other half of the inequality.

The proofs of Theorems 4.3 and 4.4 are similar to those of Theorem 4.2 and are omitted.

E. ADIABATIC INVARIANTS

ASYMPTOTIC THEORY OF HAMILTONIAN AND OTHER SYSTEMS WITH ALL SOLUTIONS NEARLY PERIODIC

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Abstract

Consider a system of N ordinary first-order differential equations in N dependent variables, and let the independent variable s not appear explicitly. Let the system depend on a small parameter ϵ and possess a formal infinite power series expansion in ϵ, and suppose that the limiting system for $\epsilon = 0$ exists and has only periodic solutions (in general not all with the same period, however) i.e., all the trajectories of points moving according to the equations form simple, closed curves. It is shown that a formal solution can be constructed involving infinite power series in ϵ and satisfying the equations over large domains of s (of order $1/\epsilon$). It is proved that the true solutions of the system exist over such domains and are asymptotically represented as $\epsilon \to 0$ by the formal solutions. The construction is based on the standard type of formal series solution (useful over bounded domains) of a "reduced" system of $N-1$ equations in $N-1$ dependent variables and with the new independent variable $\sigma = \epsilon s$; the omitted variable is essentially an angle variable θ describing the phase around the simple, closed curves. There are various interesting properties which, if possessed by the original system, are also possessed by the reduced system.

If the original system is Hamiltonian, or is even transformable into a Hamiltonian system by a formal infinite power series transformation of variables, then one can define the usual action integral $J = \oint \Sigma\, p\, dq$ to all orders; the integral is taken around the phase loop. It is proved that J is an integral (a "constant of the motion") of the system and that the Poisson bracket of θ with J is unity, both to all orders. The usefulness of this particular integral is that it is computable locally. The reduced system, after elimination of another dependent variable by means of the constancy of J, can itself be put in Hamiltonian form; if its solutions are nearly periodic, the whole procedure can be reapplied.

The present theory encompasses previous proofs of adiabatic invariance to all orders for particular systems such as the harmon-

ic oscillator [1], the nonlinear oscillator [2], the charged particle spiraling with small gyration radius and period in a given electromagnetic field [3], and the longitudinal back-and-forth motion of such a particle trapped between two "magnetic mirrors" in a weak electric field [4]. There are many other applications, not only of the result on adiabatic invariance, but more generally of the methods and results involved in obtaining the reduced system by "taking out" a relatively fast, nearly periodic variation.

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1. R. M. Kulsrud, Phys. Rev., 106, 205, (1957).

2. A. Lenard, Ann. Phys., 6, 261, (1959).

3. M. Kruskal, "The Spiraling of a Charged Particle," Rendiconti del Terzo Congresso Internazionale sui Fenomeni D'Ionizzazione nei Gas tenuto a Venezia, p. 562, Società Italiana di Fisica, Milan, (1957). Same as M. Kruskal, The Gyration of a Charged Particle, PM-S-33, NYO-7903, (1958).

4. C. Gardner, in press; also presented at this conference.

ADIABATIC INVARIANTS OF CHARGED-PARTICLE MOTION*

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Abstract

The problem of the motion of a charged particle of small mass is considered from the standpoint of perturbation theory. By a canonical transformation expressed as a power series in the mass, the Hamiltonian of the system is transformed so that Kruskal's series for the magnetic moment appears as the momentum conjugate to an ignorable coordinate. This furnishes a new proof of Kruskal's theorem on the constancy of the magnetic moment and also produces a Hamiltonian for the guiding-center motion, with two degrees of freedom. If now the particle is trapped between two magnetic mirrors in a field which varies slowly with time, a repetition of the perturbation treatment using the guiding-center Hamiltonian gives a power series which is a generalized second or longitudinal adiabatic invariant. The series is constant to all orders in the mass. Also, the dynamical system is reduced to one having <u>one</u> degree of freedom.

1. Introduction

Our subject is the motion of a charged particle of small mass in an electromagnetic field and the associated adiabatic invariants. The results are well known, but our method of treating the problem should be of interest. The method is a generalization of a simple and illuminating discussion of the adiabatic invariant of the harmonic oscillator which has been presented by Chandrasekhar.[1] The method is an adaptation of the classical methods of perturbation theory of a Hamiltonian system. The method provides a

*The work presented in this paper is supported by the AEC Computing and Applied Mathematics Center, Institute of Mathematical Sciences, New York University, under Contract AT(30-1)-1480 with the U. S. Atomic Energy Commission.

1. S. Chandrasekhar, in "The Plasma in a Magnetic Field", edited by R. Landshoff, Stanford University Press, 1958.

discussion of the validity of the adiabatic invariants which, it is felt, has advantages of clarity and simplicity. Also the method provides the basis of a discussion of the second or longitudinal adiabatic invariant which is more rigorous than discussions presented heretofore.

2. A Special Hamiltonian Formulation of the Equations of Motion

We consider a particle of mass ε and unit charge in an electromagnetic field. If $E_{\parallel}$ is the component of $\vec{E}$ parallel to $\vec{B}$ we assume

$$E_{\parallel} = O(\varepsilon) \tag{1}$$

Also we assume that the field strength B is bounded away from zero, and that $\vec{E}$, $\vec{B}$ and their derivatives are continuous and bounded. We suppose that the initial position and velocity of the particle are given independently of ε.

It is well known that for small ε the motion is compounded of three simpler motions as follows:

(a) the particle gyrates rapidly about a guiding center
(b) the guiding center moves at moderate speed along a magnetic line
(c) the guiding center drifts slowly from one magnetic line to another.

We now show how coordinates and momenta may be defined in such a way that each of the motions (a), (b), (c) is clearly associated with a particular degree of freedom.

It can be shown[2] that

$$\vec{\nabla} \cdot \vec{B} = 0$$

implies that parameters α, β (depending on the rectangular coordinates x_i and the time t) can be found so that

$$\vec{B} = \vec{\nabla}\alpha \times \vec{\nabla}\beta \tag{2}$$

Clearly α, β are constant on a magnetic line, and so a magnetic line is specified by giving α, β definite values. It follows that with the appropriate gauge we have

$$\vec{A} = \alpha \ \vec{\nabla}\beta \tag{3}$$

Let s measure arc length along a line of force. Then s, α, β may be used as geometric coordinates in place of x_i. We note that

$$-\frac{\partial}{\partial s}\left(\phi + \alpha \frac{\partial \beta}{\partial t}\right) = E_{\parallel} = O(\varepsilon) \tag{4}$$

by the assumption Eq. 1.

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2. H. Grad and H. Rubin, "Hydromagnetic Equilibria and Force-Free Fields", (Appendix I), Inst. of Math. Sci., New York Un., NYO-2358, Jan. 1959.

Now the usual Hamiltonian formulation of the equations of motion is given by

$$\bar{H} = \frac{1}{2\varepsilon} \sum_{i=1}^{3} (P_i - A_i)^2 + \phi \tag{5}$$

$$\delta \int [\sum P_i d\, x_i - \bar{H}\, dt] = \delta \int [\vec{P} \cdot d\, \vec{x} - \bar{H}\, d\, t] = 0 \tag{6}$$

The generalized coordinates here are the rectangular coordinates x_i, and the momenta P_i are given by

$$P_i = \varepsilon\, \dot{x}_i + A_i\ , \text{ or } \vec{P} = \varepsilon\, \vec{v} + \vec{A} \tag{7}$$

where $\vec{v}$ denotes the velocity vector, whose components are $\dot{x}_i$. The formulation of the equations of motion which we need is obtained by defining q_i, p_i, H by the following relations:

$$\vec{v} = \frac{\vec{P} - \vec{A}}{\varepsilon} = \frac{\vec{P} - \alpha\, \vec{\nabla}\beta}{\varepsilon} = (\vec{\nabla} s)p_2 + (\vec{\nabla}\alpha)p_1 - (\vec{\nabla}\beta)q_1 \tag{8}$$

$$\left.\begin{aligned} s &= q_2 \\ \alpha &= p_3 + \varepsilon q_1 \\ \beta &= q_3 + \varepsilon p_1 \end{aligned}\right\} \tag{9}$$

$$H = \frac{1}{\varepsilon}\, (\phi + \alpha\, \frac{\partial\beta}{\partial t}) + \frac{1}{2} \left\{ |\vec{v}|^2 + 2(\frac{\partial s}{\partial t})p_2 + 2(\frac{\partial\alpha}{\partial t})p_1 - 2(\frac{\partial\beta}{\partial t})q_1 \right\} \tag{10}$$

A straightforward calculation shows that

$$\vec{P} \cdot d\vec{x} - \bar{H}dt = p_3\, dq_3 + \varepsilon p_2 dq_2 + \varepsilon^2 p_1 dq_1 - \varepsilon H dt + d(\varepsilon p_1 p_3) \tag{11}$$

Hence, it follows from Eq. 6 that

$$\delta \int [\frac{1}{\varepsilon}\, p_3\, dq_3 + p_2\, dq_2 + \varepsilon p_1\, dq_1 - Hdt] = 0 \tag{12}$$

We now have what we may call a quasi-Hamiltonian system. Here H is given as a function of q_i, p_i, t by Eqs. 8 and 10, where the coefficients are understood to be evaluated in terms of the q_i, p_i with the aid of the relations (Eq. 9). The equations of motion which are derived from Eq. 12 are as follows:

$$\left.\begin{aligned} \dot{q}_1 &= \frac{1}{\varepsilon}\frac{\partial H}{\partial p_1} \quad , & \dot{p}_1 &= -\frac{1}{\varepsilon}\frac{\partial H}{\partial q_1} \\ \dot{q}_2 &= \frac{\partial H}{\partial p_2} \quad , & \dot{p}_2 &= -\frac{\partial H}{\partial q_2} \\ \dot{q}_3 &= \varepsilon\frac{\partial H}{\partial p_3} \quad , & \dot{p}_3 &= -\varepsilon\frac{\partial H}{\partial q_3} \end{aligned}\right\} \tag{13}$$

We see from Eqs. 8, 9 that q_1, p_1 are essentially velocity components of the gyration, and q_2, p_2 describe the motion along the line, and q_3, p_3 designate the line of force on which the guiding center is located. In this sense, then, each of the motions (a), (b), (c) is associated with a particular degree of freedom.

Note that Eqs. 8 and 9 show that each of the q's and p's is O(1). If we expand Eq. 10 in powers of ε, using Eq. 9, we obtain

$$H=\frac{1}{\varepsilon}\left(\phi+\alpha\frac{\partial\beta}{\partial t}\right)+\frac{1}{2}\left\{|\vec{v}_\perp|^2+2\left[\frac{\partial s}{\partial t}+\frac{(\vec{E}\times\vec{B})}{B^2}\cdot\vec{\nabla}s\right]p_2+p_2^{\,2}-\frac{E^2}{B^2}\right\}+O(\varepsilon) \qquad (14)$$

where now the coefficients are to be evaluated at

$$s = q_2 \ , \ \alpha = p_3 \ , \ \beta = q_3 \qquad (15)$$

and where we have used

$$E_{\parallel} = O(E).$$

Here $\vec{v}_\perp$ is defined by

$$\vec{v}_\perp = \vec{v} - \left(\frac{\vec{B}}{B}\cdot\vec{v}\right)\frac{\vec{B}}{B} - \frac{\vec{E}\times\vec{B}}{B^2} \qquad (16)$$

or

$$\begin{aligned}\vec{v}_\perp &= \vec{\nabla}\alpha\left[p_1 - \frac{(\vec{\nabla}\beta\times\vec{\nabla}s)}{B}\cdot\left(p_2\frac{\vec{B}}{B}+\frac{\vec{E}\times\vec{B}}{B^2}\right)\right] \\ &\quad - \vec{\nabla}\beta\left[q_1 - \frac{(\vec{\nabla}\alpha\times\vec{\nabla}s)}{B}\cdot\left(p_2\frac{\vec{B}}{B}+\frac{\vec{E}\times\vec{B}}{B^2}\right)\right]\end{aligned} \qquad (17)$$

Note that Eqs. 4 and 14 show that H has the form

$$H = \frac{1}{\varepsilon}H_{-1}(q_3,p_3,t) + H_o + \varepsilon H_1 + \varepsilon^2 H_2 + \dots \qquad (18)$$

where H_o, H_1, etc. depend on all the q's and p's and on t.

3. The Magnetic Moment

It is well known that the magnetic moment $v_\perp^2/B$ is an adiabatic invariant, meaning that as ε tends to zero, the quantity $v_\perp^2/B$ tends to a constant. In fact Kruskal[3] has shown how to construct a series (not necessarily convergent)

$$\mu = \mu_o + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \dots$$

where μ_k are functions of the position and velocity of the particle, whose form does not depend on the initial conditions, and where

$$\mu_o = v_\perp^2/B$$

3. M. Kruskal, The Gyration of a Charged Particle, U.S.A.E.C. Report no. PM-S-33, NYO-7903, March 12, 1958.

and such that

$$\frac{d}{dt} \sum_{k=0}^{N} \varepsilon^k \mu_k = O(\varepsilon^{N-1}) \tag{19}$$

We shall now show how this result may be derived by performing a canonical transformation on Eq.18 so that we obtain a Hamiltonian with a coordinate, which is ignorable to order ε^{N-1}. Then the conjugate momentum is constant to order ε^{N-1}, and we obtain Eq. 19.

The method consists in repeated application of a transformation which amounts to introduction of the action and angle variables corresponding to q_1, p_1, though we prefer to work with rectangular coordinates rather than with the action and angle variables, which are of the nature of polar coordinates.

We observe that if we consider H_o as function of q_1, p_1 holding all the other variables fixed, then the lines H_o = constant in the q_1 p_1 plane are nested closed curves topologically like a family of concentric circles. (In fact, Eqs. 14 and 17 show that these curves are similar concentric ellipses.) We shall try to find a canonical transformation (preserving the form Eq. 13 of the equations of motion) to $q_k^{(N)}$, $p_k^{(N)}$, $H^{(N)}$, so that as function of $q_1^{(N)}$, $p_1^{(N)}$, the curves $H^{(N)}$ = constant are indeed concentric circles, and $H^{(N)}$ has the form

$$H^{(N)} = \text{function of } (q_1^N)^2 + (p_1^N)^2;\ q_2^N, p_2^N;\ q_3^N, p_3^N;\ t$$

We cannot do this exactly, but we can to it except for an error of order ε^N; and then we obtain, using Eq. 13,

$$\frac{d}{dt} [(q_1^N)^2 + (p_1^N)^2] = O(\varepsilon^{N-1})$$

The first step is to find an area-preserving mapping of the q_1-p_1 plane into a q_1'-p_1' plane such that the lines H_o = constant in the q_1-p_1 plane get mapped into concentric circles, with center at the origin, in the q_1'-p_1' plane. Of course the mapping depends on the other variables q_2, p_2, q_3, p_3, t as parameters. Let r, θ be the polar coordinates in the q_1' p_1' plane of the image of the point q_1, p_1. Then

$$\pi r^2 = \oint p_1 \, dq_1$$
$$\theta = \frac{\int^{q_1} d\sigma / |\nabla H|}{\oint d\sigma / |\nabla H|} \tag{20}$$

where we integrate along the line H_o = constant which passes through q_1, p_1. Here σ is arc length on this line and $|\nabla H|$ is the gradient (with respect to q_1, p_1) of H. Of course πr^2 and $\theta/2\pi$ are the usual action and angle variables.

The fact that the mapping preserves area implies that

$$\oint p_1 \, dq_1 = \oint p_1' dq_1' = -\oint q_1' dp_1' \tag{21}$$

so that the line integral

$$\int [p_1'dq_1 + q_1'dp_1']$$

is independent of the path of integration in the q_1' p_1' plane, and therefore defines a function G. It is convenient to represent G as a function of q_1 and p_1', so that G is a generating function * of the transformation;

$$p_1 = \partial G/\partial q_1 \quad , \quad q_1' = \partial G/\partial p_1' \tag{22}$$

Of course, G is a function not only of q_1, p_1' but also of the parameters q_2, p_2; q_3, p_3; t.

Now we define a canonical transformation of the system Eq. 18 with the aid of the generating function

$$F = \frac{1}{\varepsilon} q_3 p_3' + q_2 p_2' + \varepsilon G(q_1, p_1'; q_2, p_2'; q_3, p_3'; t) \tag{23}$$

as follows:

$$\left.\begin{aligned} p_1 &= \frac{1}{\varepsilon}\frac{\partial F}{\partial q_1} & q_1' &= \frac{1}{\varepsilon}\frac{\partial F}{\partial p_1'} \\ p_2 &= \frac{\partial F}{\partial q_2} & q_2' &= \frac{\partial F}{\partial p_1'} \\ p_3 &= \varepsilon\frac{\partial F}{\partial q_3} & q_3' &= \varepsilon\frac{\partial F}{\partial p_3'} \end{aligned}\right\} \tag{24}$$

$$H' = H + \frac{\partial F}{\partial t} \tag{25}$$

This transformation preserves the form Eq. 13 of the equations of motion, as is seen by the easily verifiable relation

$$\left.\begin{aligned} &\frac{1}{\varepsilon} p_3 dq_3 + p_2 dq_2 + \varepsilon p_1 dq_1 - Hdt = \\ &\frac{1}{\varepsilon} p_3' dq_3' + p_2' dq_2' + \varepsilon p_1' dq_1' - H'dt \\ &+ d(F - \frac{1}{\varepsilon} q_3' p_3' - q_2' p_2' - \varepsilon q_1' p_1') \end{aligned}\right\} \tag{26}$$

which shows that Eq. 12 is preserved, and hence its consequence Eq. 13.

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* It is not always true that G is a single-valued function of q_1, p_1'; but this difficulty can be circumvented, at the cost of slightly complicating the discussion, by working with G as function of q_1, p_1.

Now the crux of the matter is that, by Eqs. 23 and 24 we see that

$$p_2 = p_2' + O(\varepsilon) \qquad q_2 = q_2' + O(\varepsilon)$$

$$p_3 = p_3' + O(\varepsilon^2) \qquad q_3 = q_3' + O(\varepsilon^2)$$

and hence by Eq. 18 we see that

$$H' = H + O(\varepsilon)$$

It follows by the construction of q_1', p_1' that H_o, as function of q_1', p_1' has the form

$$H_o = H_o'((q_1')^2 + (p_1')^2,\ q_2', p_2', q_3', p_3', t) + O(\varepsilon). \qquad (27)$$

We have made progress toward our goal. We have now that H' has the form

$$H' = \frac{1}{\varepsilon} H_o(q_3', p_3', t) + H_1'((q_1')^2 + (p_1')^2, \ldots) + \varepsilon H_1' + \ldots \quad (28)$$

The next step is to repeat the process. Namely we define q_1'', p_1'' so that the lines

$$H_o' + \varepsilon H_1' = \text{constant}$$

become concentric circles centered at the origin, when drawn in the $q_1'' - p_1''$ plane. These lines already deviate from circles only by $O(\varepsilon)$, so that q_1'', p_1'' need differ from q_1', p_1' only by $O(\varepsilon)$. The generating function G in Eq. 22 can be taken to be of the form

$$G = q_1' p_1'' + \varepsilon G_1(q_1', p_1''). \qquad (29)$$

We have then

$$F = \frac{1}{\varepsilon} q_3' p_3'' + q_2' p_2'' + \varepsilon q_1' p_1'' + \varepsilon^2 G_1(q_1', p_1'')$$

and we find

$$p_2' = p_2'' + O(\varepsilon^2) \qquad q_2' = q_2'' + O(\varepsilon^2)$$

$$p_3' = p_3'' + O(\varepsilon^3) \qquad q_3' = q_3'' + O(\varepsilon^3)$$

$$H'' = H' + O(\varepsilon^2)$$

and we obtain

$$H'' = \frac{1}{\varepsilon} H_o(q_3', p_3', t) + H_o^{(2)}((q_1'')^2 + (p_1'')^2, \ldots) + \varepsilon^2 H_2^{(2)} + \ldots$$

The process can be repeated again, working with $H_o^{(2)} + \varepsilon^2 H_2^{(2)}$, and so on. Finally we get

$$H^{(N)} = \frac{1}{\varepsilon} H_o(q_3^{(N)}, p_3^{(N)}, t) + H_o^{(N)}((q_1^{(N)})^2 + (p_1^{(N)})^2, \ldots) + O(\varepsilon^N) \quad (30)$$

And now, using the first pair of Eqs. 13 to compute $\dot{q}_1^{(N)}$ and $\dot{p}_1^{(N)}$, we readily see that

$$\frac{d}{dt} [(q_1^{(N)})^2 + (p_1^{(N)})^2] = O(\varepsilon^{N-1}) \tag{31}$$

It remains to identify the invariant here as the magnetic moment. By the area-preserving character of the mapping of q_1 p_1 into q_1', p_1' we see that $\pi[(q_1')^2 + (p_1')^2]$ is the area inside a line H = constant in the q_1,p_1 plane, and hence, by Eqs. 17 and 2 we have

$$\left.\begin{aligned} &\mu \sim (q_1^{(N)})^2 + (p_1^{(N)})^2 = (q_1')^2 + (p_1')^2 + O(\varepsilon) = \\ &\frac{v_\perp^{\;2}}{|\vec{\nabla}\alpha \times \vec{\nabla}\beta|} + O(\varepsilon) = \frac{v_\perp^{\;2}}{B} + O(\varepsilon) \end{aligned}\right\} \tag{32}$$

Hence we have the result of Kruskal which was mentioned above.

We may note that if r, θ are polar coordinates in the $q_1^{(N)}$ - $p_1^{(N)}$ plane, then $\mu = r^2$ and $-\theta/2$ are canonically conjugate, since area is given by

$$\oint p_1^{(N)} dq_1^{(N)} = \oint \frac{r^2}{2}\, d\theta = \oint \mu\, d(-\theta/2)$$

Since $H^{(N)}$ to order ε^{N-1} depends on μ but not on θ, we see that μ is constant because it is conjugate to the ignorable coordinate $-\theta/2$.

4. The Second and Third Adiabatic Invariants

We now note that except for errors of high order in ε we can replace $(q_1^{(N)})^2 + (p_1^{(N)})^2$ in Eq. 30 by the constant μ. Then we have effectively a Hamiltonian with two degrees of freedom, which describes the motion of the guiding center of the particle. Suppose now that

(1) the electric field is small
(2) the magnetic field varies slowly in time

Then the Hamiltonian of the guiding-center motion is

$$H = \frac{1}{\varepsilon}\left(\phi + \alpha \frac{\partial\beta}{\partial t}\right) + \frac{1}{2}\left\{p_2^{\;2} + \mu B\right\} + \text{higher order terms} \tag{33}$$

as we see from Eq. 14. The coefficients are to be evaluated at

$$s = q_2\ , \quad \alpha = p_3\ , \quad \beta = q_3$$

For fixed valued of q_3,p_3,t, we have

$$H = \text{const.} + \frac{1}{2}\left\{p_2^{\;2} + \mu B - \int \frac{E_\parallel}{\varepsilon}\, ds\right\} + \ldots \tag{34}$$

It is well known that it is appropriate to define a second adiabatic invariant, associated with q_2, p_2, when the lines H =

constant are nested closed curves in the q_2-p_2 plane. We see from Eq. 34 that this will be true if

$$\mu B - \int E_{\parallel}/\varepsilon \, ds$$

increases enough as one traverses a magnetic line in each direction, and has only one minimum on the line.

The method applied to the magnetic moment works here. We can show that

$$\oint p_2 dq_2 = \oint v_{\parallel} ds$$

is the first term of a series which is constant to any order in ε, and a parameter measuring the smallness of $\vec{E}$ and the slowness of the time variation. As before, any canonical transformation can be applied to q_2, p_2, and produces a result which is, to lowest order, the same as if H did not depend on q_3, p_3, t. This fact permits the carrying out of the construction.

If a second adiabatic invariant exists, then the system reduces to one having one degree of freedom, since a coordinate is ignorable. This system describes the long-term drift of the guiding center from one line of force to another. If time variations are extremely slow, and the curves H = constant in the q_3 - p_3 plane are closed, then it has been pointed out by Teller and Northrop[4] that a third adiabatic invariant may be defined. The validity of this concept follows again, by application of the methods explained herein. The appropriate definition of the third adiabatic invariant follows at once from Eq. 15. We see that

$$\oint p_3 dq_3 = \oint \alpha d\beta = \oint \alpha \vec{\nabla}\beta \cdot d\vec{s} = \oint \vec{A} \cdot d\vec{s}$$

Hence the third adiabatic invariant is the flux through the tube of lines occupied by the particle.

4. See the paper by E. Teller and T. Northrup in these Proceedings.

AN "ADIABATIC INVARIANCE THEOREM" FOR LINEAR OSCILLATORY SYSTEMS OF FINITE NUMBER DEGREES OF FREEDOM

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Abstract

The subject of this paper is the asymptotic behavior of a linear, oscillatory system in the limit where the coefficients vary slowly compared with the characteristic frequencies. Two theorems are stated and proven rigorously. The first one concerns the asymptotic expansion at times when the coefficients do not vary. The second states the sense in which the expansion is an approximation to the exact solution. Two simple special cases, given as examples, are (1) the quantum mechanical adiabatic theorem, and (2) the adiabatic invariance theorem for the harmonic oscillator.

The purpose of the following paper is to derive a general "adiabatic invariance theorem" for a linear, oscillatory system with a finite number of degrees of freedom. Such a system can be characterized by a set of coupled, ordinary, linear differential equations, which we write in matrix form:

$$\frac{dX}{dt} = AX. \tag{1}$$

Here A and X are square matrices of q^2 complex elements. The system is oscillatory if the eigenvalues of A have vanishing real parts. The system is non–degenerate if the eigenvalues are distinct. These conditions will be assumed in the following. If the matrix X(o) is non–singular, then X(t) will be non–singular for all t, and the q columns of X give q linearly independent solutions of the differential equation.

From the assumptions made about A it follows that a non–singular matrix R, and a real diagonal matrix Ω exist such that

$$R^{-1}A\,R \equiv i\,\Omega. \tag{2}$$

Furthermore, $\omega_\alpha = \Omega_{\alpha\alpha}$ are distinct ($\alpha = 1, 2, \ldots q$). The columns of R are the eigenvectors of A. R is determined only up to a transformation of the form

$$R \rightarrow RC \tag{3}$$

with a diagonal matrix C. This corresponds to the arbitrariness in the normalization of the eigenvectors. If the matrix A is independent of t, a solution of Eq. 1 is

$$X = R \exp \{i\Omega t\} \tag{4}$$

and the most general solution can be obtained by a transformation $X \rightarrow XK$ with an arbitrary constant matrix K.

Consider now the generalization of these elementary facts for the case of a slow time variation of the coefficients A. Slowness is measured by the ratio of a frequency typical of the time variation of A to the order of magnitude of its characteristic frequencies. We are thus led to investigate the asymptotic behavior of the solution of the equation

$$\frac{dX}{dt} = \lambda AX \tag{5}$$

with the real parameter $\lambda \rightarrow \infty$, while $A = A(t)$ is assumed to have a specified time dependence independent of λ. This problem was thoroughly investigated long ago by Birkhoff and Langer.[1] They considered the more general problem without the restriction that the system is oscillatory and also λ was allowed to be complex. We shall recall the essentials of this theory as applied to the simpler case considered here.

Let us assume that A, R and Ω have time derivatives of all orders, and that at the time $t = 0$ they vanish. Let us put

$$X = RY \exp \{i\lambda \int_0^t \Omega \, dt\} \tag{6}$$

and let us try to determine a formal power series in λ^{-1}

$$\sum_{n=0}^{\infty} \lambda^{-n} \, Y^{(n)}(t) \tag{7}$$

which, when substituted for Y, will make Eq. 6 satisfy the differential equation 5. In order for this to be the case Y has to satisfy the equation

$$i\lambda (\Omega Y - Y\Omega) = \frac{dY}{dt} + QY \tag{8}$$

where $Q \equiv R^{-1}(dR/dt)$. The conditions for this to be a power series identity in λ^{-1} are

$$i(\omega_\alpha - \omega_\beta) \, Y^{(o)}_{\alpha\beta} = 0 \tag{9}$$

and

$$i(\omega_\alpha - \omega_\beta) \, Y^{(n)}_{\alpha\beta} = \left(\frac{dY}{dt}^{(n-1)} + QY^{(n-1)} \right)_{\alpha\beta} \qquad (n \geq 1) \tag{10}$$

1. G. D. Birkhoff and R. Langer, Proc. Am. Acad. Arts and Sci. 58, 51 (1923). Also reprinted in "Collected Mathematical Papers" of G. D. Birkhoff.

First put $\alpha = \beta$; this gives

$$\left(\frac{dY^{(n)}}{dt} + QY^{(n)} \right)_{\alpha\alpha} = 0 \qquad (n \geq 0). \tag{11}$$

By making use of the freedom allowed by the transformation, Eq. 3, we can always make the diagonal elements of Q vanish. Then Eq. 11 allows the determination of the diagonal elements of $Y^{(n)}$ in terms of the off-diagonal ones; moreover, we set as initial conditions

$$Y^{(n)}_{\alpha\alpha}(o) = 0 \qquad (n \geq 1). \tag{12}$$

$Y^{(o)}_{\alpha\alpha}(o)$ can be left arbitrary; moreover, as will be shown below

$$Y^{(o)}_{\alpha\alpha}(t) = C_\alpha = \text{constant}. \tag{13}$$

Next take $\alpha \neq \beta$; Eq. 9 implies immediately

$$Y^{(o)}_{\alpha\beta}(t) = 0 \qquad (\alpha \neq \beta). \tag{14}$$

This, coupled with Eq. 11, substantiates the statement that the $Y^{(o)}_{\alpha\alpha}$ are time independent. Finally, the Eqs. 10 give a set of recursion relations by means of which $Y^{(n)}_{\alpha\beta}$ $(\alpha \neq \beta)$ can be expressed in terms of the elements of $Y^{(k)}$ $(0 \leq k < n)$, Q and Ω. Thus the scheme of satisfying Eq. 8 formally by Eq. 7 is complete.

We shall prove two theorems. The first one concerns an important property of the formal expansion, Eq. 7. The second one establishes the relation of the series, Eq. 7, to an exact solution of Eq. 8.

THEOREM 1. If at $t = t_o$ all time derivatives of $A(t)$ vanish, then for $t = t_o$ all time derivatives of $Y^{(n)}$ vanish and all off-diagonal elements of $Y^{(n)}$ vanish.

The proof consists of induction on n. Note that from the assumption it follows that at $t = t_o$ Q and $d\Omega/dt$ vanish together with all their time derivatives. By Eqs. 13 and 14 the theorem holds for $n = 0$. But by Eqs. 10 and 11 its validity for $n - 1$ implies its validity for n.

THEOREM 2. The series, Eq. 7, constitutes the asymptotic expansion in inverse powers of λ of that solution $Y(t, \lambda)$ of Eq. 8 which satisfies the initial condition $Y_{\alpha\beta}(0, \lambda) = C_\alpha \delta_{\alpha\beta}$. This expansion is valid uniformly over any finite time interval $0 \leq t \leq t_1$.

This means that a constant M can be found (depending on n and t_1 only, but not on t and λ) such that

$$\left| Y_{\alpha\beta}(t, \lambda) - \sum_{k=0}^{n-1} \frac{Y^{(k)}_{\alpha\beta}(t)}{\lambda^k} \right| < \frac{M}{\lambda^n} \tag{15}$$

for all α, $\beta = 1, 2, \ldots q$; all t $(0 \leq t \leq t_1)$ and all $\lambda > 0$.

To prove Theorem 2, let us fix n, and set

$$Y = \sum_{k=0}^{n} \frac{Y^{(n)}}{\lambda^n} + \frac{Z}{\lambda^n} . \tag{16}$$

This defines the matrix $Z = Z(t,\lambda)$. A simple calculation shows that Y satisfies the differential equation, Eq. 8, with the given initial condition if and only if Z is related to it by

$$Z(t,\lambda) = \int_0^t dt' \, Y(t,\lambda) \exp\{i\lambda \int_{t'}^t \Omega dt\} \, Y^{-1}(t',\lambda) \, F(t') \exp\{-i\lambda \int_{t'}^t \Omega dt\} \tag{17}$$

where

$$F(t) \equiv -Q(t) \, Y^{(n)}(t) - \frac{dY^{(n)}(t)}{dt} . \tag{18}$$

Suppose now that a positive number b_1 can be found such that

$$\begin{aligned} | Y_{\alpha\beta}(t,\lambda) | &< b_1 \\ | Y^{-1}_{\alpha\beta}(t,\lambda) | &< b_1 \end{aligned} \tag{19}$$

for $\alpha, \beta = 1, 2, \ldots q$, $0 \leq t \leq t_1$ and all $\lambda > 0$. Then if we denote by b_2 an upper bound for the absolute value of all the matrix elements of F in $0 \leq t \leq t_1$, we get from Eq. 17

$$| Z_{\alpha\beta} | < q^2 t_1 b_1^2 b_2 . \tag{20}$$

If now b_3 is an upper bound for the absolute value of the matrix elements of $Y^{(n)}$ in the interval considered, we can set

$$M = b_3 + q^2 t_1 b_1^2 b_2 \tag{21}$$

and then the inequality, Eq. 15, is satisfied. It remains to show the existence of b_1. The essential point is that the inequalities, Eq. 19, must hold for all $\lambda > 0$, even though the quantities on the left hand side depend on λ. However, from the differential equation, Eq. 8, it follows that

$$\frac{d}{dt} | Y_{\alpha\beta} |^2 = -2 \,\mathrm{Re} \sum_{\gamma} Q_{\alpha\gamma} Y_{\gamma\beta} Y^*_{\alpha\beta}, \tag{22}$$

an equation in which λ does not appear explicitly. Let us now set $N = \sum_{\alpha\beta} | Y_{\alpha\beta} |$; then by a simple chain of inequalities we derive from Eq. 22 that

$$\left| \frac{dN}{dt} \right| < q b_4 N, \tag{23}$$

where b_4 is a bound for the absolute values of all matrix elements of Q in the interval considered. But Eq. 23 implies that

$$N(t) < N(0) \exp(q b_4 t). \tag{24}$$

Thus we may take $b_1 = N(0) \exp\{q b_4 t_1\}$, and the first of the inequalities, Eq. 19, has been demonstrated. To demonstrate the second one, merely note that Y^{-1} satisfies an equation that differs from Eq. 8 only in that the order of the matrix factors in the last term is interchanged. This does not affect the argument leading to Eq. 24, and the proof of Theorem 2 is complete.

We pass on to a discussion of the significance of the results obtained. It is clear from the definition, Eq. 6, that the matrix elements of Y play the role of expansion coefficients of the solutions of the basic differential equation in terms of the eigenvectors of A, or in the usual terminology of small oscillation theory, the "normal modes." The exponential factor is the fast oscillating phase. The initial condition that Y is diagonal at $t = 0$ means that the q solutions standing in the q columns of the matrix X are those which at $t = 0$ are the q "pure modes." Theorem 2 gives an approximation to Y in terms of power series in λ^{-1} which can be calculated to any number of terms. Theorem 1 is a statement about this approximate solution. It says that <u>in regions where the coefficients of the Equation 1 smooth out to constants, the q basic solutions become pure modes.</u> This condition, imposed initially on the solution at time $t = 0$, is conserved at any other time $t = t_0$ where the conditions of Theorem 1 are satisfied (of course, only to the extent of the approximation given by the asymptotic development in λ^{-1}).

Note that Theorem 1 does not imply anything about the magnitude of the diagonal elements of the $Y^{(n)}$, i.e., the amplitudes of the normal modes. It is therefore natural to ask: Under what additional conditions is it possible to relate these diagonal elements in a simple manner to their initial values? We claim that <u>this is possible whenever some information is available about the exact solution.</u> The reason is that any exact information on Y implies some corresponding knowledge about the $Y^{(n)}$ in view Theorem 2, and, under the conditions of Theorem 1, the $Y^{(n)}$ assume the simple diagonal form, so that this information can have implications regarding the diagonal elements about which Theorem 1 makes no statement. We shall illustrate this in two notable examples.

<u>EXAMPLE 1.</u> Suppose $A = iH$, where H is a Hermitian matrix. Then Eq. 1 becomes the Schrödinger equation for a system with a finite number of non-degenerate energy eigenvalues with the Hamiltonian operator H. In such a case X, R as well as Y can be chosen to be unitary matrices, consistently with all our assumptions. But a unitary and at the same time diagonal matrix has diagonal elements of absolute value one. Hence in this case not only is $Y^{(n)}$ diagonal under the conditions of Theorem 1, but also $|Y^{(o)}_{\alpha\alpha}| = 1$ and $Y^{(n)}_{\alpha\alpha} = 0$ $(n \geq 1)$. This fact, together with $Y^{(n)}_{\alpha\beta} = 0$ $(\alpha \neq \beta,\ n \geq 0)$, is the expression of the quantum mechanical adiabatic theorem whose proof to all orders in λ^{-1} was supplied by the author of this paper.[2]

<u>EXAMPLE 2.</u> Let A be the two-rowed matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\nu^2 & 0 \end{pmatrix} \tag{25}$$

2. A. Lenard, <u>Annals of Physics</u> <u>6</u>, 261 (1959).

where $\nu = \nu(t)$ is a real, positive function. The system of equations corresponding to this choice of A is more familiar in the form of a single second order equation obtained by eliminating the variables $X_{2\alpha}$:

$$\frac{d^2X_{1\alpha}}{dt^2} + \lambda^2\nu^2 X_{1\alpha} = 0 \qquad (\alpha = 1, 2). \tag{26}$$

This is the equation for a harmonic oscillator with a time varying natural frequency $\lambda\nu(t)$. We have two exact properties of the solution. Firstly, A is a real matrix. This implies that

$$X^{-1}X^{*} = \text{constant.} \tag{27}$$

Secondly, the trace of the matrix A vanishes. This implies

$$\det X = \text{constant.} \tag{28}$$

These two exact contants of motion must be expressed in terms of Y and then use must be made of Theorem 1 in order to see what the implications are for the non-vanishing diagonal elements C_α $(\alpha = 1, 2)$. One easily checks that

$$R = \begin{pmatrix} \nu^{-\frac{1}{2}} & \nu^{-\frac{1}{2}} \\ i\nu^{\frac{1}{2}} & -i\nu^{\frac{1}{2}} \end{pmatrix} \tag{29}$$

$$\Omega = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix} \tag{30}$$

so that through Eq. 6 we can express the "constants of motion", Eq. 27 and Eq. 28, in terms of the Y. When this is done, and use is made of Theorem 1, we obtain C_1C_2 = constant, and C_1 / C_2^* = constant respectively. These imply that <u>the diagonal matrix elements of Y have constant absolute value at points where Theorem 1 applies.</u> Thus the amplitudes of the oscillations vary from one such point to another as shown by the columns of R; in particular the quantities $\nu |X_{1\alpha}|^2$ are constants (of course, only asymptotically to all orders in λ). This is the usual "adiabatic invariant" of the harmonic oscillator, whose validity for all orders in λ^{-1} was proved by R. Kulsrud.[3]

Finally, it should be pointed out that a completely analogous procedure shows that a coupled system of harmonic oscillators also has its adiabatic invariants. Not only do normal modes go over into normal modes (Theorem 1), but their amplitudes are regulated by the same inverse square root law that holds for a single harmonic oscillator.

3. R. Kulsrud, Phys. Rev. 106, 205 (1957).

PARTICLE ORBITS IN TIME DEPENDENT AXISYMMETRIC MAGNETIC FIELDS

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Abstract

The motion of a charged particle in a rapidly varying spatially uniform axisymmetric magnetic field is studied. For particular time dependences of the cyclotron frequency, $\omega(t)$, the trajectory is obtainable in closed form. If the field is varying slowly at the initial and final times a simple connection is found between the initial and final orbits. If we consider two field programs, $\omega(t)$ and $\omega_0(t)$ each of which varies rapidly, but whose ratio changes slowly, it is found that this connection is the same for both time dependences and hence defines an extended adiabatic invariant.

This paper describes an investigation of a class of problems in which the orbit of a particle in a time-dependent magnetic field is obtainable analytically. While the methods are rather straightforward and elementary, the results provide some insight into the general class of problems in which the adiabatic approximation does not apply. The origin of this work was an attempt to study the behavior of fast devices of the Scylla variety in terms of individual particle trajectories. We therefore examined the orbits of a particle in a time-dependent axisymmetric magnetic field.

Consider a cylindrical volume in which there is a magnetic field in the axial direction, which is given a function of time, but independent of coordinates. We are to obtain the orbit of a charged particle in terms of the injection conditions.

From the axial symmetry, we find at once, that the generalized angular momentum, P_θ is a constant of the motion. The equation for the radial motion is simply

$$\ddot{r} + \frac{\omega^2(t)}{4} r = \frac{C^2}{r^3} \tag{1}$$

where $\omega(t) = \frac{eB(t)}{mc}$ and $C = P_\theta/m$. The non-linear term on the right hand side of Eq. 1 represents the centripetal acceleration and can be eliminated by a transformation.

Let

$$\mu(t) = r(t) \cos\left(C\int^{t} \frac{dt'}{r^2(t')}\right) \qquad (2)$$

whose inversion is

$$r(t) = \mu(t)\left\{1 + \left(C\int^{t} \frac{dt'}{\mu^2(t')}\right)^2\right\}^{1/2} \qquad (3).$$

Then Eq. 1 reduces to

$$\ddot{\mu} + \frac{\omega^2(t)}{4}\mu = 0 \qquad (4).$$

If $\mu(t)$ is a solution of Eq. 4 satisfying the appropriate initial conditions and $\mu^+(t)$ is any other solution of Eq. 4 which is linearly independent of μ, so that the Wronskian $W\{\mu, \mu^+\}$ is non-zero, then the integral in the inversion formula, Eq. 3, is simply

$$\int^{t} \frac{dt'}{\mu^2(t')} = \frac{1}{W} \frac{\mu^+(t)}{\mu(t)} \quad .$$

By this set of transformations the orbit problem for arbitrary P_θ is reduced to that of $P_\theta = 0$.

It is well known that when the adiabatic condition is satisfied, i.e., when

$$\left|\frac{d}{dt}\omega^{-1}\right| << 1,$$

that the solutions of Eq. 4 are approximated by

$$\mu = |\omega|^{-1/2} \begin{cases} \sin(y(t)) \\ \cos(y(t)) \end{cases} \qquad (5)$$

where $$y(t) = \frac{1}{2}\int^{t} \omega(t')dt' \qquad (6).$$

It is convenient to describe the orbit in terms of an instantaneous cyclotron radius, λ, and guiding center coordinate, R. It's easy to verify that the conservation of P_θ implies

$$\omega(R^2 - \lambda^2) = \text{const.} \qquad (7)$$

Furthermore, in the adiabatic limit, combining Eqs. 3 and 5 gives an explicit, but lengthy, expression for the quantity

$$|\omega|(R^2 + \lambda^2)$$

in terms of elementary functions and constants of integration. The problem is to find these constants of integration.

In the following we confine our attention to $\omega(t)$ satisfying

$$\lim_{|t|\to\infty} \frac{d}{dt}\,\omega^{-1}(t) = 0 \qquad (8).$$

We specify the orbit in terms of the initial coordinate and velocity at $t = t_o$ and examine the trajectory at some later time t, passing to the limit $|t_o| \longrightarrow \infty$, $|t| \longrightarrow \infty$. If t and t_o have the same sign, from Eq. 8 we see that the adiabatic approximation applies throughout the motion, but if t and t_o have opposite signs this is no longer true.

Let us first consider a class of $\omega(t)$ for which Eq. 4 is integrable analytically. Taking

$$\omega(t) = kt^n \quad ; \quad n > -1$$

$$\omega(-t) = \pm\,\omega(t)$$

the condition Eq. 8 is satisfied but in the vicinity of $t = 0$ the non-adiabatic effects become very large. For this ω, the solutions of Eq. 4 are of the form

$$t^{1/2} \begin{cases} J_\nu\,(y) \\ Y_\nu\,(y) \end{cases}$$

where $\nu = (2n + 2)^{-1}$. To connect the solutions for positive and negative times one must remember to use the analytic continuation formulae for the Bessel function. Having a pair of linearly independent solutions to Eq. 4 the constants of integration are obtainable and the equation for the trajectory exhibited explicitly. While the orbit is in general very complicated and depends in detail upon the injection conditions, a rather simple result can be extracted in the limit $|t|, |t_o| \to \infty$. That is that

$$K\{\omega\} \equiv \lim_{|t_o|,|t|\to\infty} \left\langle \frac{|\omega(t_o)|\,(R^2(t) + \lambda^2(t))}{|\omega(t)|\,(R^2(t_o) + \lambda^2(t_o))} \right\rangle$$

$$= 1 + 2\,\eta \cot \pi\nu \qquad (9)$$

The symbol $\langle\ \rangle$ indicates that we are to average the initial time, t_o, over a single period, $\omega^{-1}(t_o)$. The quantity η is defined to be zero if t and t_o have the same sign, and 1 if they have opposite signs. The notation $K\{\omega\}$ indicates that K is a functional of $\omega(t)$.

Observe that all the consequences of the non-adiabatic acceleration are contained in the $2\eta \cot \pi\nu$ in Eq. 9. The parameter, ν, describing the nature of the singularity in $\frac{d}{dt}\,\omega^{-1}$. and the coefficient k does not appear.

Equations 7 and 9 combined give the mean energy of an ensemble of particles subjected to a single excursion of the field. An interesting limiting case is that of $n = 0$, $\nu = 1/2$. If $\omega(-t) = \omega(t)$ clearly nothing has happened. The choice $\omega(-t) = -\omega(t)$ means that the field changes sign stepwise at $t = 0$, and since only $|\omega|$ appears in Eq. 9, the initial and final orbits are related simply by the interchange of R and λ. In this case the average over t_0 in Eq. 9 is not needed.

When the field reversal is slower than stepwise, the particle is free to execute a more violent motion while $\dot{\omega}/\omega^2$ is small and the non-adiabatic effect is increased. A physically interesting case is that of $n = 1$ (the field changes sign linearly) so that the right hand side of Eq. 9 is equal to 3.

Our problem now is to find out whether these results are generalizable to other, more complicated forms of $\omega(t)$. Suppose that for some $\omega_0(t)$ the quantity $K\{\omega_0\}$ is known. Suppose this ω_0 is in some sense a good approximation to $\omega(t)$. How well is $K\{\omega\}$ approximated by $K\{\omega_0\}$?

We formulate this problem more precisely as follows. Consider an $\omega_0(t)$ satisfying condition Eq. 8, and such that ω_0^{-1} is continuously differentiable except at one point which we conveniently choose to be the origin of time. Consider another $\omega(t)$ such that $\omega(t)/\omega_0(t)$ is continuously differentiable at $t = 0$ and $\lim_{|t|\to\infty} \omega(t)/\omega_0(t) = \text{constant}$ (not necessarily the same constant for positive and negative t). Let the ratio ω_0/ω be a slowly varying function of time, whose rate of change is described by a "slowness parameter", ε. We seek to express $K\{\omega\} - K\{\omega_0\}$ as a power series in ε.

This problem can be studied by an extension of the method given by Kulsrud[1]. The analysis is slightly more complicated, but nothing fundamentally new need be added and only the result will be given here. We find that $K\{\omega\} - K\{\omega_0\}$ vanishes to as many orders in ε as ω_0/ω has continuous derivatives.

This enables us to generalize the concept of an adiabatic invariant. Ordinarily one seeks "local" properties of the orbit, such as energy, which is invariant under slow changes in ω. Here we allow ω to vary rapidly and find that another ω_0 provides a good model for ω, in the sense that an "integral property" of the orbit, K, is the same for ω and ω_0 if ω_0/ω is slowly varying. With choice of the model function ω_0 = const the conventional type adiabatic invariance is recovered.

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1. R. Kulsrud, Phys. Rev. 106, 205 (1957).

F. SHOCK WAVES

MAGNETO-HYDRODYNAMIC SHOCK STRUCTURE WITHOUT COLLISIONS*

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Abstract

The problem of proving the existence of a magneto-hydrodynamic shock without collisions consists of finding a solution to two Boltzmann equations without collision terms and two Maxwell equations. For a classical shock structure the solution would lead from one constant state at large distances in one direction (the state ahead) to a different constant state at large distances in the other direction (the state behind). However, a solution which leads from a constant state ahead to a periodic state behind may be interpreted as part of a shock if the entropy in some sense increases. Such solutions have been shown to exist theoretically if the mass ratio is very small and the characteristic wavelength is kept fixed. This length is the geometric mean of the distances the ions and electrons move forward in a complete change of phase in a constant magnetic field. The prescribed distribution function for the ions ahead of the shock is a Maxwellian cut-off at some speed. Such solutions have now been computed for various values of the Alfven number, pressure and cut-off speed ahead of the shock. For certain cut-off speeds there is no "shock". For other values of the cut-off speed, in a certain range of Mach numbers, there is a shock in the sense described above with a large change in the mean magnetic field. The mean magnetic field, for example, may be increased through the shock by 75 percent and the oscillation is about 20 percent of the final value. Therefore entropy increase is a large fraction of the theoretical maximum.

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A shock structure analysis of a medium strength steady shock involving no collisions was presented last year.[1] It used the fact that the ions are much heavier than the electrons. This report presents the results of numerical computations based on the analysis.

First we pose the full problem and discuss the earlier analysis. One wants to find a flow which leads from one constant state (given) very far in one direction to another constant state very far in the other direction. The width of the transition should be such that it is reasonable to approximate it by a sharp discontinuity.

We consider only the case in which the magnetic field at large distances is perpendicular to the direction of flow. The variables of the flow are the two distribution functions $f_{\pm}$ for the ions and electrons, the electric field $\vec{E}$ and magnetic field $\vec{H}$. There is no dependence on time and x is the space variable. Then $f_{\pm}$ are functions of x and the two components of velocity u, v.

The equations governing the flow are the two Boltzmann equations, without collisions, for $f_{\pm}$, and Maxwell's equations which involve integrals of $f_{\pm}$ in the charge density and current. The boundary conditions given at $x = -\infty$, are $E_x = 0$, $E_y \neq 0$, $E_z = 0$, $H_x = 0$, $H_y = 0$, $H_z \neq 0$. $f_+ = f_+^{\infty}$, $f_- = f_-^{\infty}$ when $f_{\pm}^{\infty}$ are any isotropic distributions consistent with the constant state.

Nothing is known about the solution of the full problem but there are four alternative possibilities. a) No solution exists. b) There is a shock solution i. e. a solution which approaches a constant state as $x \rightarrow \infty$ which is different from the given state at $-\infty$. c) The solution is a pulse, i. e. the state at $+\infty$ is the same as the state at $-\infty$. d) The solution does not tend to a constant state at $+\infty$.

Within the approximation based on a large ion to electron mass ratio, there is no strict shock solution b). For a certain range of parameters a) occurs and for other ranges either alternatives c) or d). Alternative c) bears no relation to a shock. But in d) one finds, at $+\infty$, fields that are periodic in x; the pressure oscillates around a mean pressure much higher than at $x = -\infty$. If the oscillations are small not much energy is in the vibrations. We may interpret such solutions as shock solutions. In calculating them we find that they correspond to a low β and a big increase in ion temperature.

For the approximation we set $\sqrt{m_-/m_+} = \epsilon$, introduce as a fixed length the geometric mean of the distances an ion or electron moves forward

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1. C. S. Morawetz, "Steady State Shock Model," Controlled Thermonuclear Conference, Washington, D.C., February, 1958. TID-7558. See also a summary in C. S. Gardner et al., "Hydromagnetic Shock Waves in High-Temperature Plasmas," Papers Presented at the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva, September 1958, NYO-2538, Institute of Math. Sci., NYU, Jan. 1959.

in a gyroperiod and normalize the fields so that the magnetic field is of order 1 but the charge separation field is of order $1/\epsilon$.

The functions $f_{\pm}$ can be found as asymptotic series in ϵ. For electrons the procedure follows the method of Gardner-Kruskal et al. For ions the expansion is in some respects more straightforward. However we must take $f_{+}^{\infty} = \exp - K((1-u)^2 + v^2)$ for $(1-u)^2 + v^2 < R^2$, and $f_{+}^{\infty} = 0$ for $(1-u)^2 + v^2 > R^2$. Here K is a dimensionless constant which measures the ratio of thermal energy to the energy of translation, $K \sim 1/2\beta$. The mean velocity at $-\infty$ is normalized to 1 and R, the cut-off radius, is a number < 1 which is chosen differently for different problems. (There is one other dimensionless parameter A the ratio of mean speed to Alfven speed at $-\infty$.)

For any problem we take A, K fixed and study the solution to lowest order in ϵ for different values of R. For R small all ions pass through without turning around and the flow returns to the state it was in at $-\infty$ (case c)). As we increase R we reach a critical value beyond which there are always some ions that turn around. Thus theoretically we get solutions when the mean velocity and magnetic field change monotonically. Because of the characteristic length the electrons are adiabatic and the ions are heated.

If we let $R \to 1$ we can again show that our approximation breaks down. That is, it is not valid for ions that just barely escape turning around in the constant field at $-\infty$. It is therefore necessary in computing to have R big enough so some ions turn around and small enough so approximation is valid.

This means that if we want to picture our cut-off Maxwellian distribution as an approximation to a true Maxwellian, then the distribution function must be very narrow, i.e., K is big or β small.

The computations were made with two additional assumptions: 1) The total charge density is zero -- there is virtually no loss of accuracy. 2) Where the change in magnetic field is small and the electric field is of order 1 particle paths have been computed as if the fields were constant.

With A = 1.4, K = 4 we find that the field quantities are very sensitive to the value of R, see Figures 1, 2, 3. In fact solutions cease to exist for R > .45.

With A = 1.4, K = 75 we find that as R increases we approach a region where the fields vary little with R, Figures 4, 5, 6. Here we get finite oscillations about markedly increased mean values. We may expect this solution to be a good approximation to a true solution of the full problem. The initial increase of H with no change in η (essentially the charge separation field) takes place in a kind of boundary layer of width $O(1/\epsilon)$ ahead of the main transition region. In fact this boundary layer plays a fundamental role in analyzing the motion of the ions.

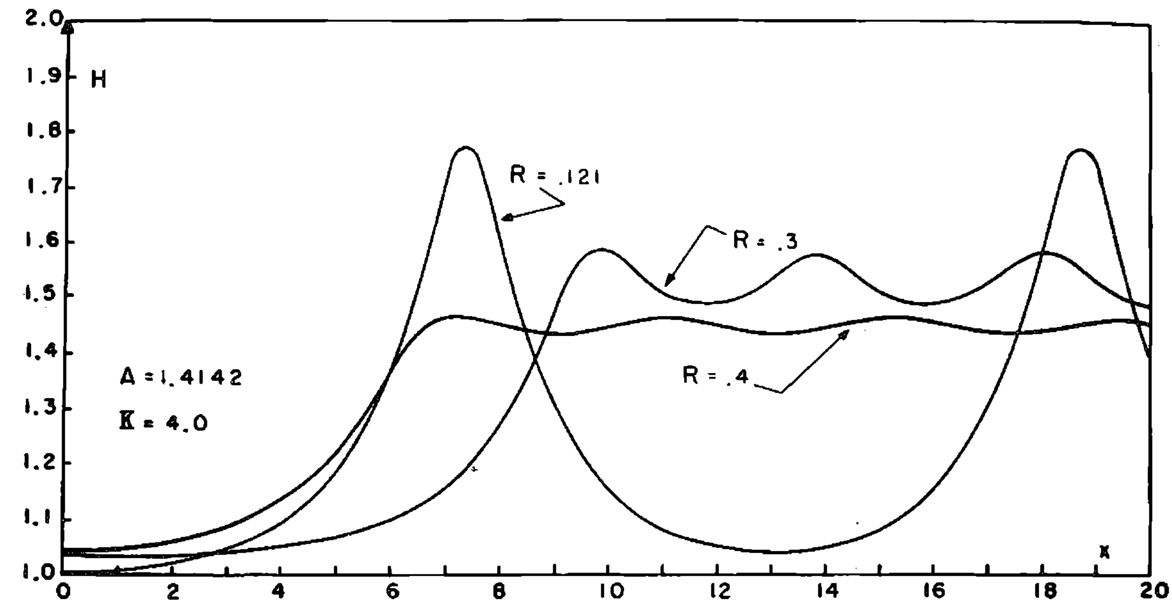

FIGURE 1

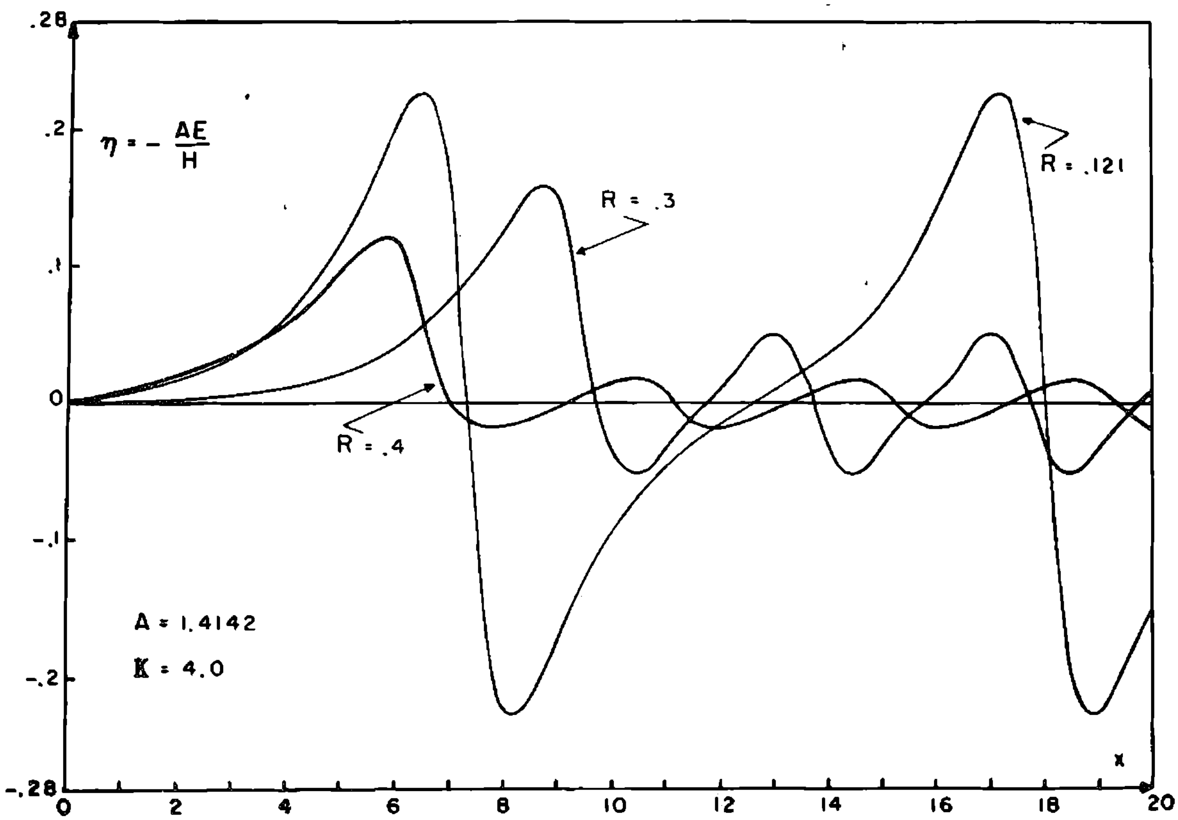

FIGURE 2

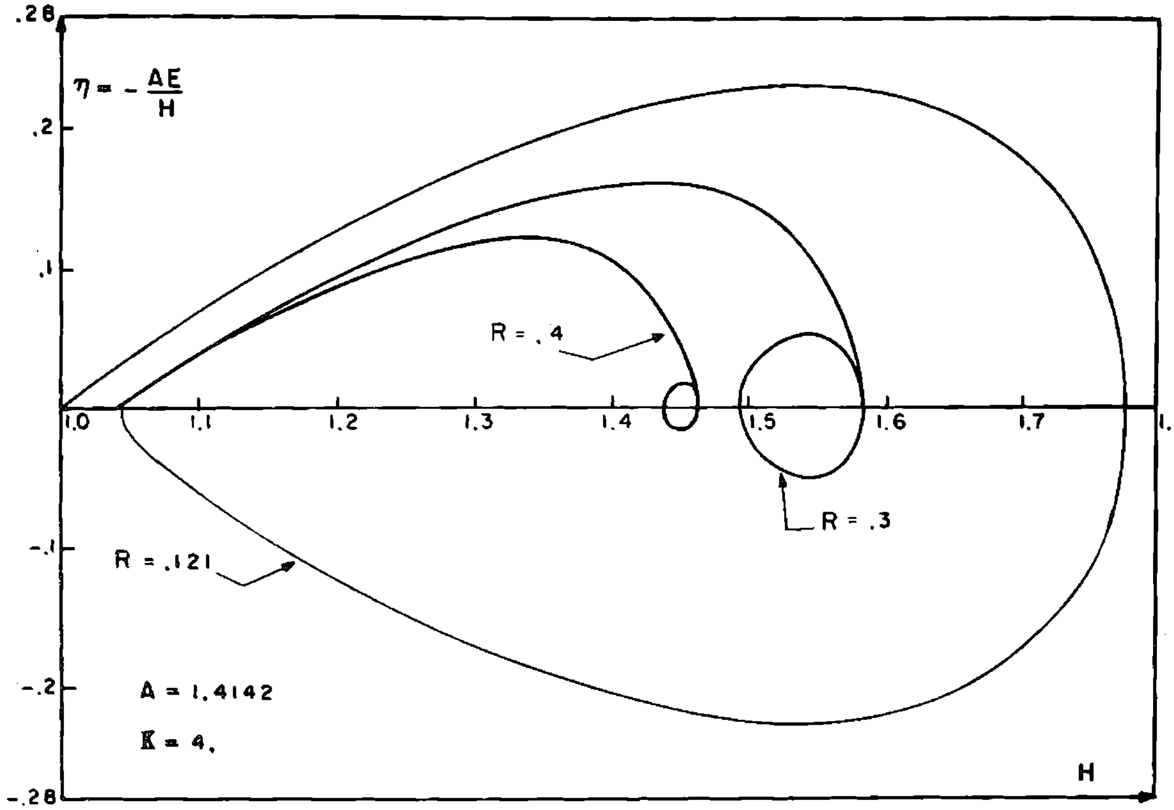

FIGURE 3

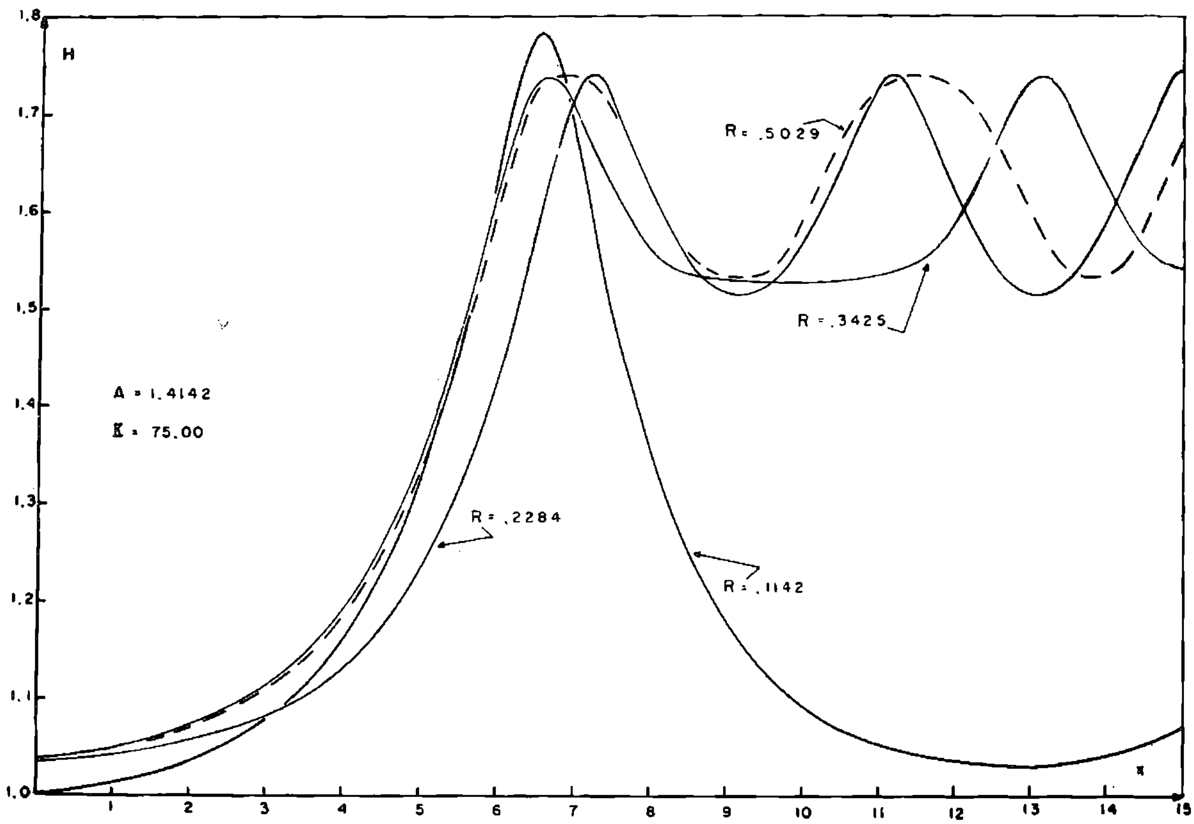

FIGURE 4

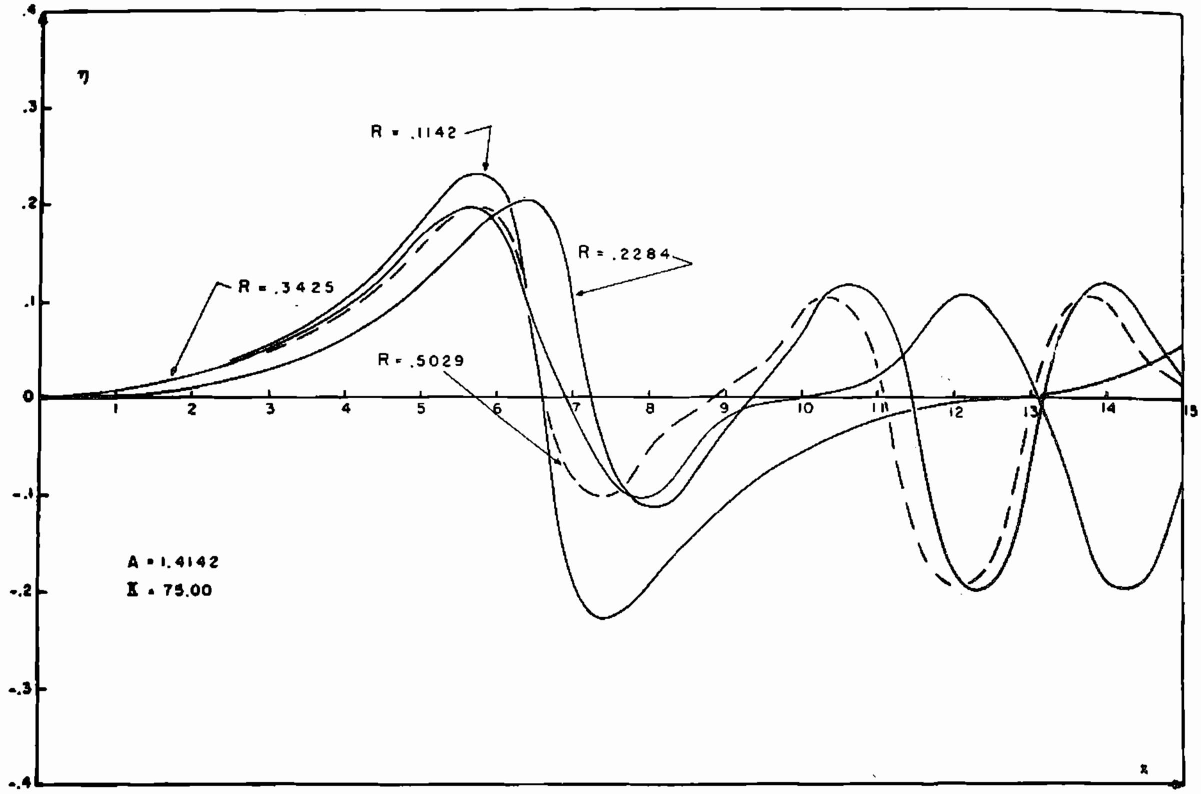

FIGURE 5

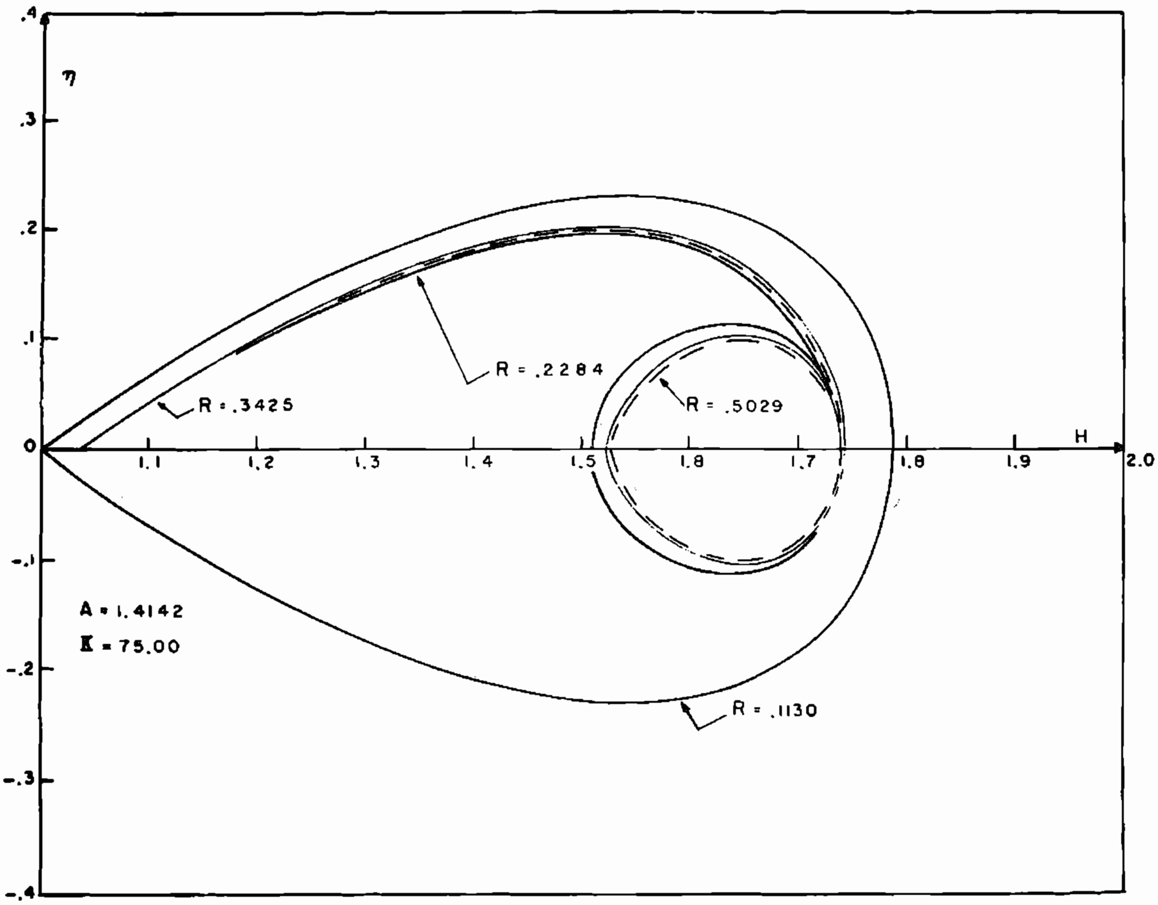

FIGURE 6

G. ADDITIONAL PAPERS

INCREASED DISPERSION AND RESISTIVITY IN A NONSTEADY PLASMA*

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In thermodynamic equilibrium, charge fluctuations in a plasma give rise to electric field fluctuations, i.e. they excite plasma oscillations. These oscillations are damped quickly if the wave length is comparable to the Debye length or is smaller, but large wave lengths are only slightly damped. This is compensated by the fact that the excitation is small for large wave lengths. These fluctuating electric fields or, equivalently, encounters between particles give rise to the dynamical friction and dispersion coefficients in the Fokker-Planck equation.

In a non-equilibrium state, there are external (as distinguished from thermal charge fluctuation) sources of excitation. These give rise to additional plasma oscillations superposed on the thermal background. These oscillations can be the consequence of instabilities or merely the result of complicated initial conditions or external influences brought to bear on the plasma. The large wave length components (larger than the Debye length) can be very slowly damped and can therefore be present with much larger amplitude than is expected from thermodynamic considerations.

To estimate the effect of such externally induced plasma oscillations on the value of the dynamical friction and dispersion coefficients requires an estimate of the magnitude of the fluctuating field and of the correlation time as seen by a single particle. For example, a coherent, precisely periodic plasma oscillation will produce no net effect on a constant-speed particle since one half-cycle exactly cancels the next. Let us assume that a certain number, ν, of plasma oscillation periods is the correlation time; i.e. the fluctuating electric field is coherent over ν wave lengths. In one half-cycle, the

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impulse on a particle of mass M is

$$\delta(Mv) = eE/\Omega \tag{1}$$

where E is a representative amplitude of the electric field and Ω is the plasma frequency,

$$\Omega^2 = ne^2/\varkappa_o m \tag{2}$$

(m is the mass of the electron.) In one correlation time, $\tau = \nu/\Omega$, the net, uncancelled impulse has the same order as in Eq. 1. After a time large compared to τ, say $N\tau$, the root-mean-square expected impulse has a magnitude $\sqrt{N}$ times the value given in Eq. 1,

$$\delta v = \sqrt{N}\,\frac{eE}{M\Omega} \quad . \tag{3}$$

We set $\delta v = \sqrt{kT/M}$ and solve for the value of N,

$$N_o = \frac{\Omega^2 MkT}{e^2E^2} = \frac{M}{m}\,\frac{nkT}{\varkappa_o E^2} \quad . \tag{4}$$

This is the number of correlation times required to produce a deflection of 90° in a representative particle; $N_o\tau$ is an equivalent collision or thermalization time, and $N_o\tau\sqrt{kT/M}$ is an equivalent mean-free-path,

$$L^* = N_o\tau\sqrt{kT/M} = \nu\sqrt{\frac{M}{m}}\;\frac{(kT)^2}{de^2E^2} \quad . \tag{5}$$

Here d is the Debye length,

$$d^2 = \frac{kT}{m\Omega^2} = \frac{\varkappa_o kT}{ne^2} \quad . \tag{6}$$

We compare L^* with the mean-free-path for thermalization of the particles M among themselves which can be expressed as

$$L = \frac{\varkappa_o kTd^2}{e^2\log\Lambda} \quad , \tag{7}$$

and obtain

$$\frac{L^*}{L} = \nu\,\log\Lambda\,\sqrt{\frac{M}{m}}\,\left(\frac{kT}{\varkappa_o E^2 d^3}\right) \quad . \tag{8}$$

The denominator of the last factor is the electrostatic energy in a Debye sphere.

The maximum electric field that one can expect to arise in a plasma has the order

$$eEd = kT \tag{9}$$

which yields the minimum value for L^*/L,

$$\frac{L^*}{L} = \nu \log \Lambda \sqrt{\frac{M}{m}} \frac{1}{nd^3} \tag{10}$$

A reasonable estimate for ν might be 10; $\log \Lambda$ is on the order of 15; in a high temperature plasma nd^3 (the number of particles in a Debye sphere) may be on the order of 10^6. We conclude that, for deuterons ($M/m \sim 3600$), L^* might be as small as L, signifying that the extraneous electric fields are as important as the thermal fluctuations, whereas for electrons ($M \sim m$), L^* can be even much smaller than L. Alternatively, L^* becomes comparable to L for electrons when the fluctuating electric field is as small as one per cent of the maximum that can be encountered.

We can draw the following conclusions. In a boundary layer or sheath where we know that the steady value of E is comparable to the "maximum" value in Eq. 9, even a very slight unsteadiness of the sheath can increase markedly the dispersion (and therefore the resistivity). In the interior of the plasma, it is possible for moderate amplitude electric field oscillations to have the same effect. In the early stages of mixing of crossed or reversed magnetic fields, it is likely that a large part of the current is confined to a narrow region in which there is also a very large electric field. This situation could lead to an excessively large resistivity of the plasma itself. Since this phenomenon is inherently inaccessible to precise theoretical computation, great caution should be exercised in employing resistivity measurements as a diagnostic tool.

STABILITY OF RADIOFREQUENCY PLASMA CONFINEMENT

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Abstract

The stability of confinement of a conducting fluid by rf electromagnetic fields is investigated statically in plane geometry by assuming that only time averaged field pressures need be considered. It is found that the set of perturbation wave numbers can be divided into stable and unstable intervals. In general, the confinement is stable if the wave length of the boundary deformations is sufficiently short. Similar results are obtained if various steady magnetic fields are also postulated to be present.

The problem and solution discussed below are part of a general study of the possibility of plasma confinement by electromagnetic fields in which the frequency is high enough to require consideration of displacement currents in the field equations.[1] The same problem has also been treated by E. Weibel.[2]

Perhaps the simplest problem of the type mentioned in the title is that exemplified by Fig. 1. The plasma is assumed to be a perfectly conducting fluid, leading to a sharp interface between plasma and field regions. If an electromagnetic field consisting of transverse standing waves is excited in the space between the fluid and the wall, and if the configuration is constrained to be invariant under lateral translations, it is easy to show that an equilibrium interface position exists, provided only that the internal energy of the fluid increases indefinitely with decreasing volume. Here one needs an adiabatic theorem for resonant cavities which states that the quantity

$$\frac{\text{total field energy}}{\text{field frequency}}$$

remains constant under slow deformations of the cavity walls.[3] Clearly, the equilibrium interface may be located at $z=0$.

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1. J. W. Butler, Bull. Am. Phys. Soc., Ser. II, 4, 152 (1959) (Abstract).

2. Erich S. Weibel, On the Confinement of Plasma by Standing Electromagnetic Waves, ARL-57-1009 (1957).

3. F. E. Borgnis and C. H. Papas, "Electromagnetic Waveguides", Handbuch der Physik 16, p. 412 (1958).

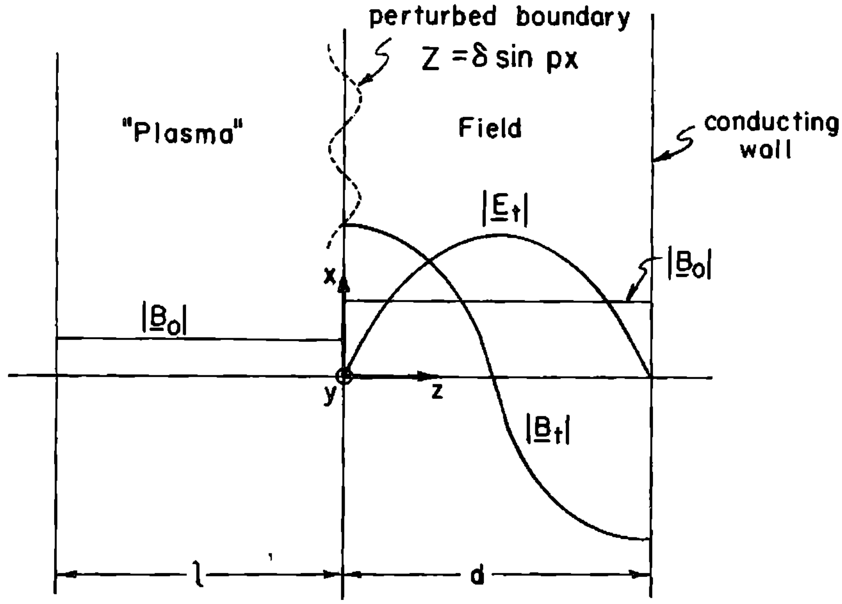

Figure 1.

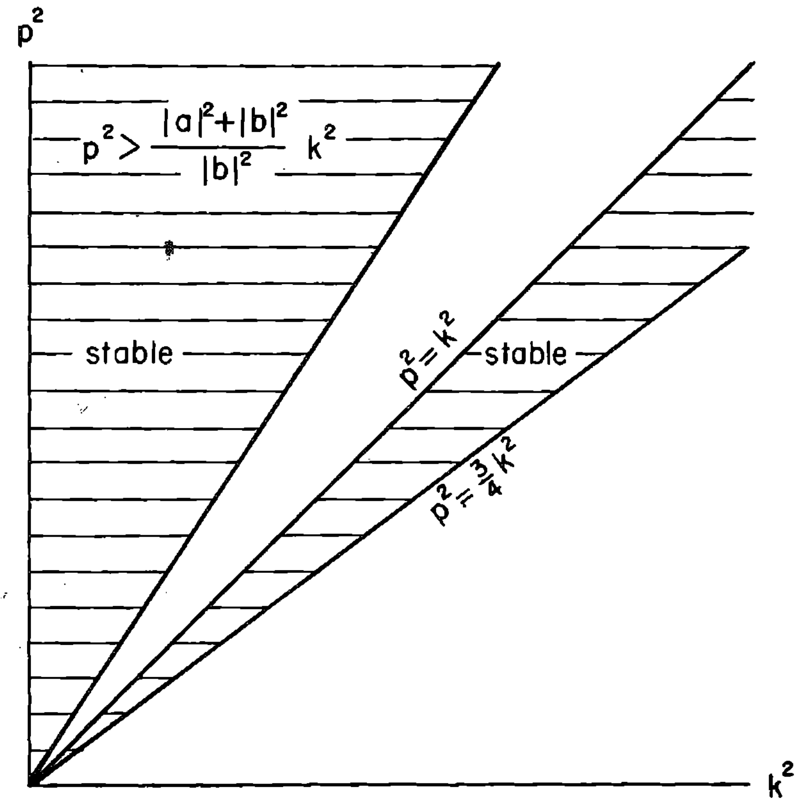

Figure 2.

The next step is to remove the translation invariance in the x direction and see if the equilibrium state is thereby altered. For a first order treatment this is sufficient, since as will be seen, the field can be rotated to any desired orientation. Accordingly, take the new interface to be the corrugated surface $z = \delta \sin px$; the normal is evidently of the form $\underline{n} = (O(\delta),0,1)$. Components of vectors are listed in conventional x,y,z order.

To calculate the perturbed field to first order in δ, form the scalar function

$$e_t = \sin kz - \delta k \sin px \frac{\sinh \varkappa(d-z)}{\sinh \varkappa d},$$

where $\varkappa^2 = p^2 - k^2$ and $k = \pi/d = \omega/c$, the angular field frequency divided by light speed. The transverse electric phasor is then constructed as

$$\underline{E}_t = (ae_t,\ be_t,\ E_z),$$

in which a and b are arbitrary complex numbers and E_z is given by

$$E_z = -\int_0^z du(a\frac{\partial}{\partial x} e_t) - \delta \frac{k}{\varkappa} \cos px \cosh \varkappa d = O(\delta),$$

coming from the divergence condition $\mathrm{div}\underline{E}_t = 0$. This electric vector then satisfies the required field equations

$$\nabla^2\underline{E}_t + k^2\underline{E}_t = 0,$$

$$\underline{n} \times \underline{E}_t = O(\delta^2) \text{ at } z = \delta \sin px,$$

$$\underline{n} \times \underline{E}_t = 0 \qquad \text{at } z = d,$$

and, of course, $\mathrm{div}\underline{E}_t = 0$. It is possible to construct the perturbed field without changing the wave number k because the perturbation considered does not change the volume and hence, to first order in δ, leaves the resonant frequency unaltered.

The remainder of the calculation is routine. The transverse magnetic phasor is computed from Maxwell's equation

$$\underline{B}_t = \frac{i}{\omega} \mathrm{curl}\underline{E}_t,$$

and the field pressure acting on the deformed surface is evaluated as

$$P_{rf} = \left\langle \frac{1}{4\mu} \underline{B}_s \cdot \underline{B}_s^* \right\rangle,$$

where $\underline{B}_s$ is the value of $\underline{B}_t$ at the surface $z = \delta \sin px$, the symbol < > denotes a time average, and * is used to indicate complex conjugation. The electric field makes only a second order contribution to the stress tensor at the surface, since the normal and tangential components of $\underline{E}_t$ are respectively of order δ and δ^2. When evaluated in detail, the expression for P_{rf} turns out to be

$$P_{rf} = \frac{|a|^2+|b|^2}{4\mu c^2}\left[1 + 2\delta \sin px \left\{\frac{1}{\varkappa}\left(\frac{|b|^2}{|a|^2+|b|^2}p^2 - k^2\right)\coth \varkappa d\right\} + O(\delta^2)\right] \qquad (1)$$

which indicates first order stability whenever the quantity in { } is positive.

The resulting stability diagram is as shown in Fig. 2; the hatched zones contain stable parameter values.

In the particular case of circular field polarization, one therefore has instability in the p intervals defined by

$$k^2 < p^2 < 2k^2 \tag{2}$$

and

$$p^2 < \frac{3}{4} k^2.$$

For elliptical polarization, it is clear that actual stability information can only be obtained by orienting the major axis of the electric polarization ellipse in the x direction, since this is the least stable alignment and therefore the most favorable for the growth of surface disturbances. This is accomplished by assigning a real value to a and setting $b = ira$, $(0 \leq r \leq 1)$, changing the inequality 2 to read

$$k^2 < p^2 < \frac{1+r^2}{r^2} k^2.$$

It is seen that the upper stability zone in the diagram evaporates as $r \to 0$; the lower one, however, is unaffected. On the line $p^2 = k^2$, there is apparently no first order solution to the field equations.

Stability can be restored in the plane polarized situation by introducing a steady transverse magnetic field into the cavity region which is perpendicular to the rf magnetic field. Indeed, this causes the resultant field vector at the plasma surface to oscillate in direction, which might be expected to have a stabilizing influence similar to that of rotation in the pure rf case.

By letting $k \to 0$ in Eq. 1 and suitably reinterpreting the numbers a and b, one obtains the pressure distribution due to the perturbed dc field as

$$P_{dc} = \frac{b_x^2 + b_y^2}{2\mu}\left[1 + 2\delta p \sin px \frac{b_x^2}{b_x^2 + b_y^2} \coth pd + O(\delta^2)\right], \tag{3}$$

in which b_x and b_y are the components of the undisturbed magnetic field. Returning to Eq. 1, let a and b have real values and specify the direction of the resulting alternating magnetic vector by $s^2 = b^2/(a^2 + b^2)$; orthogonality of the rf and dc fields then requires $b_x^2/(b_x^2 + b_y^2) = 1 - s^2$. Making these substitutions and adding Eqs. 1 and 3 yields the stability criterion

$$\frac{1}{\chi}(s^2p^2 - k^2)\coth \chi d + f^2(1-s^2)p \coth pd > 0, \tag{4}$$

wherein f^2 is the ratio of dc to rf field pressure. The stability chart resulting from this inequality is quite similar to Fig. 4 for reasonable parameter values. Thus, taking $f^2 = 1$ and approximating $\coth \chi d \sim \coth pd \sim 1$, it is found that 4 is again satisfied for $p^2 > 2k^2$, whatever the value of s. Inspection also shows that, by making f^2 larger, the instability zones can be reduced in size but not eliminated; the inequality can always be reversed by choosing p^2 slightly greater than k^2.

If steady magnetic fields inside the "plasma" are allowed, it becomes necessary to inquire into the kinematics of the fluid-field mixture. Furthermore, for a finite thickness, boundary conditions must be applied at the interface $z = -\ell$ (see Fig. 1), which makes further analysis of this particular configuration somewhat artificial as applied to actual plasma confinement. One moderately interesting problem, however, is obtained by restricting the unperturbed magnetic field inside the plasma to have only a z component. The results in this case show that, for an infinite thickness $(\ell = \infty)$, the field component b_z has no influence on stability, although there is a favourable effect if the thickness is finite and the region $z < -\ell$ is assumed to be a rigid conductor.